

QUANTUM MATRIX BALL: THE BERGMAN KERNEL

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1 Introduction

A study of q -analogues for bounded symmetric domains, in particular, for quantum matrix balls, was started in [6, 7]. This work continues studying these 'balls' and presents the associated Bergman kernels.

Everywhere below $q \in (0, 1)$, $m \leq n$, $N = m + n$.

We assume knowledge of the results of [7] and keep the notation of that work.

2 $\text{Pol}(\text{Mat}_{mn})_{q,y} \simeq \text{Pol}(X)_{q,x}$

The $*$ -algebra $\text{Pol}(\text{Mat}_{mn})_q$, a quantum analogue of polynomial algebra on the space of matrix, was described in [7]. This algebra was defined in terms of the generators z_a^α , $\alpha = 1, \dots, m$; $a = 1, \dots, n$, and the commutation relations

$$z_a^\alpha z_b^\beta - q z_b^\beta z_a^\alpha = 0, \quad a = b \quad \& \quad \alpha < \beta, \quad \text{or} \quad a < b \quad \& \quad \alpha = \beta, \quad (2.1)$$

$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha = 0, \quad \alpha < \beta \quad \& \quad a > b, \quad (2.2)$$

$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha - (q - q^{-1}) z_a^\beta z_b^\alpha = 0, \quad \alpha < \beta \quad \& \quad a < b, \quad (2.3)$$

$$(z_b^\beta)^* z_a^\alpha = q^2 \cdot \sum_{a', b'=1}^n \sum_{\alpha', \beta'=1}^m R_{ba'}^{b' a'} R_{\beta \alpha}^{\beta' \alpha'} \cdot z_{a'}^{\alpha'} (z_{b'}^{\beta'})^* + (1 - q^2) \delta_{ab} \delta^{\alpha\beta}, \quad (2.4)$$

with δ_{ab} , $\delta^{\alpha\beta}$ being the Kronecker symbols, and

$$R_{ij}^{kl} = \begin{cases} q^{-1} & , \quad i \neq j \quad \& \quad i = k \quad \& \quad j = l \\ 1 & , \quad i = j = k = l \\ -(q^{-2} - 1) & , \quad i = j \quad \& \quad k = l \quad \& \quad l > j \\ 0 & , \quad \text{otherwise} \end{cases}$$

The subalgebra generated by $(z_a^\alpha)^*$, $\alpha = 1, \dots, m$; $a = 1, \dots, n$, is denoted by $\mathbb{C}[\overline{\text{Mat}}_{mn}]_q$, and the subalgebra generated by z_a^α , $\alpha = 1, \dots, m$; $a = 1, \dots, n$, is denoted by $\mathbb{C}[\text{Mat}_{mn}]_q$. Obviously, $\text{Pol}(\text{Mat}_{mn})_q = \mathbb{C}[\text{Mat}_{mn}]_q \mathbb{C}[\overline{\text{Mat}}_{mn}]_q$.

Our work [7] provides an embedding of the $*$ -algebra $\text{Pol}(\text{Mat}_{mn})_q$ into a $*$ -algebra of functions on a quantum principal homogeneous space. Remind the definition of that $*$ -algebra.

Consider the well known algebra $\mathbb{C}[SL_N]_q$ of regular functions on the quantum group SL_N . Its generators are t_{ij} , $i, j = 1, \dots, N$, and the complete list of relations includes the relations similar to (2.1) – (2.3) and the equality $\det_q T = 1$. (Here $\det_q T$ is a q -determinant of the matrix $T = (t_{ij})_{i,j=1,\dots,N}$:

$$\det_q T \stackrel{\text{def}}{=} \sum_{s \in S_N} (-q)^{l(s)} t_{1s(1)} \cdot t_{2s(2)} \cdots t_{Ns(N)}, \quad (2.5)$$

with $l(s) = \text{card}\{(i, j) \mid i < j \text{ \& } s(i) > s(j)\}$.)

Note that q -minors of T are defined similarly to (2.5):

$$t_{IJ}^{\wedge k} \stackrel{\text{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} t_{i_1 j_{s(1)}} \cdot t_{i_2 j_{s(2)}} \cdots t_{i_k j_{s(k)}},$$

with $I = \{(i_1, i_2, \dots, i_k) \mid 1 \leq i_1 < i_2 < \dots < i_k \leq N\}$, $J = \{(j_1, j_2, \dots, j_k) \mid 1 \leq j_1 < j_2 < \dots < j_k \leq N\}$.

In [7] the two $*$ -algebras $\text{Pol}(\tilde{X})_q = (\mathbb{C}[SL_N]_q, *)$ and $\mathbb{C}[SU_N]_q = (\mathbb{C}[SL_N]_q, \star)$ have been considered, with the involutions $*$ and \star being given by

$$\begin{aligned} t_{ij}^* &= (-q)^{j-i} t_{\{1, \dots, \widehat{i}, \dots, N\} \{1, \dots, \widehat{j}, \dots, N\}} \\ t_{ij}^* &= \text{sign}((i - m + \tfrac{1}{2})(n - j + \tfrac{1}{2})) t_{ij}^*. \end{aligned} \quad (2.6)$$

They are called the algebra of polynomial functions on a quantum principal homogeneous space of SU_{nm} and the algebra of regular functions on the quantum group SU_N , respectively. (The latter $*$ -algebra is well known (see [1])).

We follow [7] in introducing the notation $t = t_{\{1, 2, \dots, m\} \{n+1, n+2, \dots, N\}}$, $x = tt^*$.

The following lemma is deducible from (2.6) and a general formula of Ya. Soibelman [5], [1, p. 432] for the involution \star .

Lemma 2.1 *Let $\text{card}(J) = m$, $J^c = \{1, 2, \dots, N\} \setminus J$, $l(J, J^c) = \text{card}\{(j', j'') \in J \times J^c \mid j' > j''\}$. Then*

$$(t_{\{1, 2, \dots, m\} J}^{\wedge m})^* = (-1)^{\text{card}(\{1, 2, \dots, n\} \cap J)} (-q)^{l(J, J^c)} t_{\{m+1, m+2, \dots, N\} J^c}^{\wedge n}. \quad (2.7)$$

Corollary 2.2 $tt^* = t^*t$.

Consider a localization $\text{Pol}(\tilde{X})_{q,x}$ of the $*$ -algebra $\text{Pol}(\tilde{X})_q$ with respect to the multiplicative system x, x^2, x^3, \dots . (The $*$ -algebra $\text{Pol}(\tilde{X})_{q,x}$ has no zero divisors. It is derivable from $\text{Pol}(\tilde{X})_q$ via adding a selfadjoint element x^{-1} : $xx^{-1} = x^{-1}x = 1$, $(x^{-1})^* = x^{-1}$.)

A localization $\mathbb{C}[SL_N]_{q,t}$ of the algebra $\mathbb{C}[SL_N]_q$ with respect to the multiplicative system t, t^2, \dots is defined in a similar way. Evidently, $\mathbb{C}[SL_N]_{q,t} \hookrightarrow \text{Pol}(\tilde{X})_{q,x}$ since $t^{-1} = t^*x^{-1} = x^{-1}t^*$ by corollary 2.2.

The embedding of \ast -algebras $\mathcal{J} : \text{Pol}(\text{Mat}_{mn})_q \hookrightarrow \text{Pol}(\tilde{X})_{q,x}$ mentioned above is given by $\mathcal{J} : z_a^\alpha \mapsto t^{-1} t_{\{1,2,\dots,m\}J_{a\alpha}}^{\wedge m}$, with $J_{a\alpha} = \{n+1, n+2, \dots, N\} \setminus \{N+1-\alpha\} \cup \{a\}$.

A crucial point in what follows will be such element $y \in \text{Pol}(\text{Mat}_{mn})_q$ that $\mathcal{J}y = x^{-1}$. Our immediate intention is to construct this y .

We need a notation for q -minors of the matrix $(z_a^\alpha)_{\alpha=1,\dots,m; a=1,\dots,n}$. Suppose $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq m$, $1 \leq a_1 < a_2 < \dots < a_k \leq n$. Set up

$$z_{\{a_1, a_2, \dots, a_k\}}^{\wedge k \{ \alpha_1, \alpha_2, \dots, \alpha_k \}} \stackrel{\text{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} z_{a_1}^{\alpha_{s(1)}} z_{a_2}^{\alpha_{s(2)}} \dots z_{a_k}^{\alpha_{s(k)}}.$$

In the proof of the next statement we use the fact that $\mathbb{C}[SL_N]_q$ and $\mathbb{C}[SL_N]_{q,t}$ are $U_q \mathfrak{sl}_N^{\text{op}} \otimes U_q \mathfrak{sl}_N$ -module algebras, $\mathbb{C}[\text{Mat}_{mn}]_q$ is a $U_q \mathfrak{sl}_N$ -module algebra, and the embedding \mathcal{J} is a morphism of $U_q \mathfrak{sl}_N$ -modules. When considering $U_q \mathfrak{sl}_N$ - and $U_q \mathfrak{sl}_N^{\text{op}}$ -modules, we use, together with the elements K_i , $i = 1, \dots, N-1$, the operators H_i , $i = 1, \dots, N-1$, introduced in [7]. (The relationship between those is $K_i = q^{H_i}$, $i = 1, \dots, N-1$.)

Lemma 2.3 *Let $k \in \mathbb{N}$. There exists $c(q, k)$ such that for all $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq m$, $1 \leq a_1 < a_2 < \dots < a_k \leq n$, in the algebra $\mathbb{C}[SL_N]_{q,t}$, one has*

$$\mathcal{J} z_{\{a_1, a_2, \dots, a_k\}}^{\wedge k \{ m+1-\alpha_k, m+1-\alpha_{k-1}, \dots, m+1-\alpha_1 \}} = c(q, k) t^{-1} t_{\{1, 2, \dots, m\}J}^{\wedge m}, \quad (2.8)$$

with $J = \{n+1, n+2, \dots, N\} \setminus \{n+\alpha_1, n+\alpha_2, \dots, n+\alpha_k\} \cup \{a_1, a_2, \dots, a_k\}$.

Proof. Consider the linear span of all $z_{\{a_1, a_2, \dots, a_k\}}^{\wedge k \{ m+1-\alpha_k, m+1-\alpha_{k-1}, \dots, m+1-\alpha_1 \}}$ and the linear span of all $t^{-1} t_{\{1, 2, \dots, m\}J}^{\wedge m}$, with k being fixed. They are both free modules over the subalgebra $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m \subset U_q \mathfrak{sl}_N$. Since \mathcal{J} is a morphism of $U_q \mathfrak{sl}_N$ -modules, it suffices to prove (2.8) in the special case as follows:

$$\mathcal{J} z_{\{n-k+1, n-k+2, \dots, n\}}^{\wedge k \{ m-k+1, m-k+2, \dots, m \}} = c(q, k) t^{-1} t_{\{1, 2, \dots, m\}J_k}^{\wedge m},$$

with $J_k = \{n-k+1, n-k+2, \dots, n, n+k+1, n+k+2, \dots, N\}$.

Let $\mathbb{F} \subset \mathbb{C}[SL_N]_q$ be a subalgebra generated by $\{t_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq N\}$ and $\mathbb{F}_t \subset \mathbb{C}[SL_N]_{q,t}$ a subalgebra generated by the same elements as above and t^{-1} .

It is easy to prove that $f = \mathcal{J} z_{\{n-k+1, n-k+2, \dots, n\}}^{\wedge k \{ m-k+1, m-k+2, \dots, m \}}$ belongs to \mathbb{F}_t and is a solution of the following system of homogeneous linear equations:

$$(E_i \otimes 1)f = (F_i \otimes 1)f = (H_i \otimes 1)f = 0, \quad i = 1, 2, \dots, m-1, \quad (2.9)$$

$$(H_m \otimes 1)f = 0,$$

$$(1 \otimes F_j)f = 0, \quad j \neq n,$$

$$(1 \otimes H_j)f = \begin{cases} -f & , \quad j = n \pm k \\ 0 & , \quad j \notin \{n, n-k, n+k\} \end{cases}.$$

Since $t^{-1} t_{\{1, 2, \dots, m\}J_k}^{\wedge m}$ satisfies all the above equations, it suffices to prove that the space of solutions $f \in \mathbb{F}_t$ of this system is one-dimensional. Arguing just as in the proof of [7, lemma 8.2], we obtain the following results. The subalgebra of solutions $f \in \mathbb{F}_t$ of (2.9) is generated by $t^{-1}, t_{\{1, 2, \dots, m\}I}^{\wedge m}$, $\text{card } I = m$. The subalgebra \mathbb{F}_{inv} of solutions of (2.9) together with $(H_m \otimes 1)f = 0$, is generated by ratios of quantum minors $t^{-1} t_{\{1, 2, \dots, m\}I}^{\wedge m}$, $\text{card } I = m$,

and the subalgebra $\mathbb{F}_{\text{prim}} = \{f \in \mathbb{F}_{\text{inv}} \mid (1 \otimes F_j)f = 0, j \neq m\}$ is generated by ratios of quantum minors $t^{-1}t_{\{1,2,\dots,m\}J_l}^{\wedge m}$, $J_l = \{n-l+1, n-l+2, \dots, n, n+l+1, n+l+2, \dots, N\}$, $l \leq m$.

What remains is to consider the linear span \mathfrak{h}' of $1 \otimes H_j$, $j \neq n$, and to elaborate the linear independence in $(\mathfrak{h}')^*$ of the weights of generators in \mathbb{F}_{prim} :

$$(1 \otimes H_j) (t^{-1}t_{\{1,2,\dots,m\}J_l}^{\wedge m}) = \begin{cases} -t^{-1}t_{\{1,2,\dots,m\}J_l}^{\wedge m} & , \quad j = n \pm l \\ 0 & , \quad j \notin \{n, n-l, n+l\} \end{cases} \quad \blacksquare$$

The action of the subalgebra $U_q \mathfrak{sl}_N \hookrightarrow U_q \mathfrak{sl}_N^{\text{op}} \otimes U_q \mathfrak{sl}_N$ is to be referred to in the sequel more extensively than that of the subalgebra $U_q \mathfrak{sl}_N^{\text{op}} \hookrightarrow U_q \mathfrak{sl}_N^{\text{op}} \otimes U_q \mathfrak{sl}_N$. Just as in [7], we write ξf instead of $(1 \otimes \xi)f$ in all the cases where this could not lead to a confusion.

Apply the relations $F_n z_a^\alpha = q^{1/2} \delta_{an} \delta_{\alpha m}$, $H_n z_b^\beta = 0$ for $b \neq n$ and $\beta \neq m$ (see [7]) to get

$$F_n z_{\{n-k+1,\dots,n\}}^{\wedge k \{m-k+1,\dots,m\}} = q^{1/2} z_{\{n-k+1,\dots,n-1\}}^{\wedge k-1 \{m-k+1,\dots,m-1\}}.$$

It follows from $\Delta(F_n) = F_n \otimes K_n^{-1} + 1 \otimes F_n$, $F_n(t^{-1}) = 0$ that

$$\begin{aligned} F_n (t^{-1}t_{\{1,2,\dots,m\}\{n-k+1,\dots,n,n+k+1,\dots,N\}}^{\wedge m}) &= t^{-1}F_n t_{\{1,2,\dots,m\}\{n-k+1,\dots,n,n+k+1,\dots,N\}}^{\wedge m} = \\ &= q^{1/2} t^{-1} t_{\{1,2,\dots,m\}\{n-k+1,\dots,n-1,n+1,n+k+1,\dots,N\}}^{\wedge m}. \end{aligned}$$

Hence, $c(q, k) = c(q, k-1) = \dots = c(q, 1) = 1$, and thus we have proved

Proposition 2.4 *Let $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq m$, $1 \leq a_1 < a_2 < \dots < a_k \leq n$, $J = \{n+1, n+2, \dots, N\} \setminus \{n+\alpha_1, n+\alpha_2, \dots, n+\alpha_k\} \cup \{a_1, a_2, \dots, a_k\}$. Then*

$$\mathcal{J} z_{\{a_1, a_2, \dots, a_k\}}^{\wedge k \{m+1-\alpha_k, m+1-\alpha_{k-1}, \dots, m+1-\alpha_1\}} = t^{-1} t_{\{1,2,\dots,m\}J}^{\wedge m}$$

Turn back to a construction of such $y \in \text{Pol}(\text{Mat}_{mn})_q$ that $\mathcal{J}y = x^{-1}$. It follows from (2.5), (2.7) that

$$\sum_{\substack{J \subset \{1,\dots,N\} \\ \text{card}(J)=m}} (-1)^{\text{card}(\{1,2,\dots,n\} \cap J)} t_{\{1,2,\dots,m\}J}^{\wedge m} (t_{\{1,2,\dots,m\}J}^{\wedge m})^* = 1. \quad (2.10)$$

The following is due to proposition 2.4, (2.10), and the injectivity of \mathcal{J} .

Theorem 2.5 *There exists a unique element $y \in \text{Pol}(\text{Mat}_{mn})_q$ such that $\mathcal{J}y = x^{-1}$. It is given explicitly by*

$$y = 1 + \sum_{k=1}^m (-1)^k \sum_{\{J' \mid \text{card}(J')=k\}} \sum_{\{J'' \mid \text{card}(J'')=k\}} z^{\wedge k J'_{J''}} \left(z^{\wedge k J'_{J''}} \right)^*,$$

with $J' \subset \{1, 2, \dots, m\}$, $J'' \subset \{1, 2, \dots, n\}$.

EXAMPLE 2.6. In the case of quantum ball in \mathbb{C}^n ($m = 1$), one has $z_a = z_a^1$, $y = 1 - \sum_{a=1}^n z_a z_a^*$.

Corollary 2.7 For all $\alpha = 1, \dots, m$, $a = 1, \dots, n$, one has

$$z_a^\alpha y = q^{-2} y z_a^\alpha, \quad (z_a^\alpha)^* y = q^2 y (z_a^\alpha)^*.$$

Note that $f \in \text{Pol}(\text{Mat}_{mn})_q$, $f \neq 0$ imply $yf \neq 0$, $fy \neq 0$ because of the injectivity of the map $\mathcal{J} : \text{Pol}(\text{Mat}_{mn})_q \rightarrow \text{Pol}(\tilde{X})_{q,x}$. Hence $\text{Pol}(\text{Mat}_{mn})_q \hookrightarrow \text{Pol}(\text{Mat}_{mn})_{q,y}$.

The work [7] introduces a subalgebra $\text{Pol}(X)_q$ of all $U_q \mathfrak{sl}_m \times \mathfrak{sl}_n$ -invariants of $\text{Pol}(\tilde{X})_q$. Let $\text{Pol}(X)_{q,x}$ and $\text{Pol}(\text{Mat}_{mn})_{q,y}$ be respectively localizations of the $*$ -algebras $\text{Pol}(X)_q$ and $\text{Pol}(\text{Mat}_{mn})_q$ with respect to the multiplicative systems $x^\mathbb{N}$, $y^\mathbb{N}$. Now theorem 2.5 and [7, proposition 8.1] provide a 'canonical' isomorphism

Proposition 2.8 $\text{Pol}(\text{Mat}_{mn})_{q,y} \xrightarrow[\mathcal{J}]{} \text{Pol}(X)_{q,x}$.

(The injectivity of \mathcal{J} is evident. In fact, it follows from $\mathcal{J}\left(\sum_{k=0}^M y^{-k} f_k\right) = 0$, $f_0, f_1, \dots, f_M \in \text{Pol}(\text{Mat}_{mn})_q$, that $\mathcal{J}\left(\sum_{k=0}^M y^{M-k} f_k\right) = 0$, and so $\sum_{k=0}^M y^{M-k} f_k = 0$, $\sum_{k=0}^M y^{-k} f_k = 0$.)

REMARK 2.9. In some contexts the generators z_a^α become inconvenient and are to be replaced by $z_{\alpha a} = (-q)^{\alpha-1} z_a^{m+1-\alpha}$, $\alpha = 1, \dots, m$, $a = 1, \dots, n$. (This passage could be treated as drawing down the 'Greek' index via the 'tensor' $\varepsilon = (-q)^{\alpha-1} \delta_{\alpha+\beta, m+1}$.)

Proposition 2.10 Consider the matrix $Z = (z_{\alpha a})_{\alpha=1, \dots, m, a=1, \dots, n}$. In the matrix algebra with entries from $\mathbb{C}[SL_N]_{q,t}$ one has

$$\mathcal{J}(Z) = T_{12}^{-1} T_{11}, \quad (2.11)$$

with $\mathcal{J}(Z) = (\mathcal{J}(z_{\alpha a}))$, $T_{11} = (t_{\alpha a})$, $T_{12} = (t_{\alpha n+\beta})$, $\alpha, \beta = 1, \dots, m$, $a = 1, 2, \dots, n$.

Proof. Let $T = (t_{ij})_{i,j=1, \dots, m}$ and $\det'_q T = \sum_{s \in S_m} (-q)^{-l(s)} t_{s(m)m} t_{s(m-1)m-1} \cdots t_{s(1)1}$. We are about to prove that in $\mathbb{C}[\text{Mat}_{mn}]_q$ one has

$$\det'_q T = \det_q T. \quad (2.12)$$

The relation $\det'_q T = \text{const}(q) \det_q T$ follows from the $U_q \mathfrak{sl}_m$ -invariance of $\det'_q T$, $\det_q T$, if one also takes into account that the spaces of homogeneous degree m invariants are one-dimensional. To compute the constant $\text{const}(q)$, it suffices to pass to the quotient algebra with respect to the bilateral ideal generated by t_{ij} , $i \neq j$.

Now (2.11) is derivable from (2.12) and the explicit form of T_{12}^{-1} :

$$(T_{12}^{-1})_{\alpha\beta} = (\det_q T_{12})^{-1} \cdot (-q)^{\alpha-\beta} \det_q((T_{12})_{\beta\alpha}), \quad \alpha, \beta = 1, \dots, m$$

(Here, just as in the classical case $q = 1$, $(T_{12})_{\beta\alpha}$ is a matrix derived from T_{12} by discarding the line β and column α .) ■

REMARK 2.11. In the classical limit ($q = 1$) one has

$$y = 1 + \sum_{k=1}^m (-1)^k \text{tr}(Z^{\wedge k} (Z^*)^{\wedge k}) = 1 + \sum_{k=1}^m (-1)^k \text{tr}((ZZ^*)^{\wedge k}) = \det(1 - ZZ^*).$$

(These relations are evident since their proof reduces to considering the special case $z_{\alpha a} = \lambda_\alpha \delta_{\alpha a}$, $\lambda_\alpha \in \mathbb{C}$.)

3 Hardy-Bergman spaces

Remind some results of [7]. An extension $\text{Fun}(\mathbb{U})_q$ of the covariant $*$ -algebra $\text{Pol}(\text{Mat}_{mn})_q$ was produced there via adding to the list of its generators $\{z_a^\alpha\}$ of such element f_0 that $f_0 = f_0^2 = f_0^*$, and $(z_a^\alpha)^* f_0 = f_0 z_a^\alpha = 0$ for all $\alpha = 1, \dots, m$, $a = 1, \dots, n$.

Let $D(\mathbb{U})_q = \text{Fun}(\mathbb{U})_q f_0 \text{Fun}(\mathbb{U})_q$ be the $*$ -algebra of finite functions in the quantum ball, and $\mathcal{H} = \text{Fun}(\mathbb{U})_q f_0 = \mathbb{C}[\text{Mat}_{mn}]_q f_0$. It follows from the explicit formulae for the action of $U_q \mathfrak{sl}_N$ in $\text{Fun}(\mathbb{U})_q$ (see [7]) that $D(\mathbb{U})_q$ is a $U_q \mathfrak{su}_{nm}$ -module algebra, and \mathcal{H} is a $U_q \mathfrak{b}_+$ -module algebra. (Here $U_q \mathfrak{b}_+ \subset U_q \mathfrak{sl}_N$ is a subalgebra generated by $K_j^{\pm 1}$, E_j , $j = 1, \dots, N-1$). We follow [7] in denoting the 'natural' representations on $\text{Fun}(\mathbb{U})_q$ and $U_q \mathfrak{b}_+$ in \mathcal{H} by Θ and Γ respectively. In section 9 of that work an explicit formula for the invariant integral was obtained:

$$\int_{\mathbb{U}_q} f d\nu = \text{tr}(\Theta(f)\Gamma(e^{h\tilde{\rho}})), \quad f \in D(\mathbb{U})_q, \quad (3.1)$$

with $\tilde{\rho} = \frac{1}{2} \sum_{j=1}^{N-1} j(N-j)H_j$, $h > 0$, and the operators $\Gamma(H_j)$ in \mathcal{H} are determined by $q = e^{-h/2}$, $\Gamma(K_j^{\pm 1}) = q^{\pm \Gamma(H_j)}$, $j, \dots, N-1$. (We follow an agreement of [7] in restricting to considering only those $U_q \mathfrak{sl}_N$ -modules which admit well defined actions of the 'elements' H_j , $X_j^+ = E_j q^{-\frac{1}{2}H_j}$, $X_j^- = q^{\frac{1}{2}H_j} F_j$, $j = 1, \dots, N-1$.)

Among the main results of [7] one should mention, in particular, the positivity of the scalar product in \mathcal{H} defined by

$$(\psi_1, \psi_2)_{f_0} = \psi_2^* \psi_1, \quad \psi_1, \psi_2 \in \mathcal{H}. \quad (3.2)$$

The positivity of the scalar product (3.2) implies the positivity of the invariant integral (3.1) (see [7]).

Remind (see [7]) that $\mathbb{C}[\text{Mat}_{mn}]_q = \bigoplus_{k=0}^{\infty} \mathbb{C}[\text{Mat}_{mn}]_{q,k}$, $\mathbb{C}[\text{Mat}_{mn}]_{q,k} = \{f \mid \deg f = k\}$, $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$, $\mathcal{H}_k = \mathbb{C}[\text{Mat}_{mn}]_{q,k} f_0$. The equality $y f_0 = f_0$ and the commutation relations $y z_a^\alpha = q^2 z_a^\alpha y$, $\alpha = 1, \dots, m$, $a = 1, \dots, n$ imply

Lemma 3.1 *For all $k, \lambda \in \mathbb{Z}_+$ one has*

$$\Theta(y)^\lambda|_{\mathcal{H}_k} = q^{2k\lambda} I. \quad (3.3)$$

The non-integral powers $\Theta(y)^\lambda \in \text{End}(\mathcal{H})$ of the operator $\Theta(y)$ are defined by (3.3).

Lemma 3.2 *For $\lambda > N-1$ one has*

$$\text{tr}(\Theta(y)^\lambda \Gamma(e^{h\tilde{\rho}})) = \prod_{j=0}^{n-1} \prod_{k=0}^{m-1} (1 - q^{2(\lambda+1-N)} q^{2(j+k)})^{-1}$$

Proof. Equip the linear span \mathfrak{h} of the 'elements' H_j , $j = 1, 2, \dots, N-1$, with a scalar product

$$(H_i, H_j) = \begin{cases} 2 & , \quad i = j = 0 \\ -1 & , \quad |i - j| = 1 \\ 0 & , \quad |i - j| > 1 \end{cases}.$$

Let $\{\alpha_j\}$ be the standard basis of simple roots in \mathfrak{h}^* . Then $\alpha_j(\check{\rho}) = (H_j, \check{\rho}) = 1$ for all $j = 1, \dots, N-1$, and the relation

$$\begin{aligned}\check{\rho}z_n^m &= \frac{1}{2}((m+1)(n-1)H_{n-1} + (m-1)(n+1)H_{n+1} + mnH_n)z_n^m = \\ &= \frac{1}{2}(-(m+1)(n-1) - (m-1)(n+1) + 2mn)z_n^m = z_n^m\end{aligned}$$

implies

$$\check{\rho}z_a^\alpha = (N+1-a-\alpha)z_a^\alpha, \quad \alpha = 1, \dots, m, \quad a = 1, \dots, n. \quad (3.4)$$

The vectors $\{(z_1^1)^{k_{11}} \dots (z_n^m)^{k_{mn}} f_0\}$, $k_{a\alpha} \in \mathbb{Z}_+$, $\alpha = 1, \dots, m$, $a = 1, \dots, n$, form a basis in \mathcal{H} , and, by a virtue of (3.4), one has

$$\Gamma(e^{h\check{\rho}})((z_1^1)^{k_{11}} \dots (z_n^m)^{k_{mn}} f_0) = q^{-2\sum k_{a\alpha}(N+1-a-\alpha)}((z_1^1)^{k_{11}} \dots (z_n^m)^{k_{mn}} f_0). \quad (3.5)$$

What remains is to apply the definition of the operator $\Theta(y)^\lambda$:

$$\mathrm{tr}(\Theta(y)^\lambda \Gamma(e^{h\check{\rho}})) = \sum_{k_{11}=0}^{\infty} \dots \sum_{k_{mn}=0}^{\infty} q^{2\sum_{a=1}^n \sum_{\alpha=1}^m (\lambda - (N+1-a-\alpha))k_{a\alpha}} \quad (3.6)$$

and to sum the geometric progressions in (3.6). ■

Proposition 3.3 *For any $f \in \mathrm{Fun}(\mathbb{U})_q$, $\Theta(f)$ is a bounded operator in the pre-Hilbert space \mathcal{H} .*

Proof. By a virtue of [7, remark 8.6], it suffices to consider the case $f \in \mathrm{Pol}(\mathrm{Mat}_{mn})_q$. In the work alluded above, a $*$ -representation Π was constructed; it was also shown to be unitary equivalent to the $*$ -representation Θ . Thus, the inequality $\|\Theta(f)\| < \infty$ follows from $\|\Pi(f)\| < \infty$ (see the appendix). ■

By proposition 3.3, for $\lambda > N-1$ one has a well defined linear functional

$$\int_{\mathbb{U}_q} f d\nu_\lambda \stackrel{\mathrm{def}}{=} C(\lambda) \mathrm{tr}(\Theta(f) \Theta(y)^\lambda \Gamma(e^{h\check{\rho}})), \quad f \in \mathrm{Fun}(\mathbb{U})_q,$$

with

$$C(\lambda) = \prod_{j=0}^{n-1} \prod_{k=0}^{m-1} (1 - q^{2(\lambda+1-N)} q^{2(j+k)}). \quad (3.7)$$

Proposition 3.4 *For all $\lambda > N-1$, the linear functional $\int_{\mathbb{U}_q} f d\nu_\lambda$ on the $*$ -algebra $\mathrm{Fun}(\mathbb{U})_q$ is positive, and $\int_{\mathbb{U}_q} 1 d\nu_\lambda = 1$.*

Proof. The latter relation follows from lemma 3.2. To verify the positivity of $\int_{\mathbb{U}_q} f d\nu_\lambda$, it suffices to observe that the $*$ -representation Θ of $\text{Fun}(\mathbb{U})_q$ is faithful, and the bounded operator $C(\lambda)\Theta(y)^\lambda \Gamma(e^{h\tilde{p}}) = C(\lambda)\Gamma(e^{h\tilde{p}})\Theta(y)^\lambda$ is positive. ■

Consider a completion $L^2(d\nu_\lambda)_q$ of the vector space $\text{Fun}(\mathbb{U})_q$ with respect to the norm $\|f\|_\lambda = \left(\int_{\mathbb{U}_q} f^* f d\nu_\lambda \right)^{1/2}$. The closure $L_a^2(d\nu_\lambda)_q$ of the linear subvariety $\mathbb{C}[\text{Mat}_{mn}]_q$ in the Hilbert space $L^2(d\nu_\lambda)_q$ will be called the Hardy-Bergman space.

There exists a very useful approach in which the $*$ -algebra $\text{Pol}(\text{Mat}_{mn})_q$ is treated as a q -analogue of the Weil algebra, and the operators $(1 - q^2)^{-1/2}T(z_a^\alpha)$, $(1 - q^2)^{-1/2}T(z_a^\alpha)^*$ as q -analogues of creation and annihilation operators respectively. In this context, the following result becomes a q -analogue of the Stone-von-Neumann theorem.

Theorem 3.5 *There exists a faithful irreducible $*$ -representation of $\text{Pol}(\text{Mat}_{mn})_q$ by bounded operators in a Hilbert space. This representation is unique up to unitary equivalence.*

Proof. By proposition 3.3, there exists a well defined $*$ -representation $\overline{\Theta}$ of $\text{Pol}(\text{Mat}_{mn})_q$ in a completion $\overline{\mathcal{H}}$ of the pre-Hilbert space \mathcal{H} . This $*$ -representation is faithful by [7, proposition 8.8]. Furthermore, if a bounded linear operator A commutes with all the operators $\overline{\Theta}(f)$, $f \in \text{Pol}(\text{Mat}_{mn})_q$, then, in particular $\overline{\Theta}(y)A = A\overline{\Theta}(y)$. Hence $Af_0 = af_0$ for some $a \in \mathbb{C}$ since $\mathbb{C}f_0$ is an eigenspace of $\overline{\Theta}(y)$. It follows that $A = aI$. That is, $\overline{\Theta}$ is irreducible. What remains is to demonstrate a uniqueness of the faithful irreducible $*$ -representation.

Let T be a faithful irreducible $*$ -representation of $\text{Pol}(\text{Mat}_{mn})_q$ by bounded linear operators in a Hilbert space. The same idea as in [10] can be used to prove that the non-zero spectrum of the selfadjoint operator $T(y)$ is discrete. Consider some eigenvector v of $T(y)$ associated to a largest modulus eigenvalue. By a virtue of corollary 2.7, $T((z_a^\alpha)^*)v = 0$, $\alpha = 1, 2, \dots, m$, $a = 1, 2, \dots, n$. It is easy to show that the kernels of the linear functionals $(\overline{T(f)}v, v)$, $(\Theta(f)f_0, f_0)$ on $\text{Pol}(\text{Mat}_{mn})_q$ are just the same subspace $\bigoplus_{(j,k) \neq (0,0)} \mathbb{C}[\text{Mat}_{mn}]_{q,j} \mathbb{C}[\overline{\text{Mat}_{mn}}]_{q,-k}$. Thus $(T(f)v, v) = \text{const}(\Theta(f)f_0, f_0)$, $\text{const} > 0$, and hence the map $f_0 \mapsto (\text{const})^{-1/2}v$ admits an extension up to a unitary map which intertwines the representations $\overline{\Theta}$ and T . ■

4 Distributions

It will be proved below that the orthogonal projection P_λ in the Hilbert space $L^2(d\nu_\lambda)_q$ onto the subspace $L_a^2(d\nu_\lambda)_q \subset L^2(d\nu_\lambda)_q$ is an integral operator. Our principal intention is find the kernel of this integral operator.

We follow [7] with beginning the construction of distributions in the quantum matrix ball via a completion operation.

Impose three topologies in $\text{Pol}(\text{Mat}_{mn})_q$ and prove their equivalence.

Associate to each element $\psi \in D(\mathbb{U})_q$ a linear functional $l_\psi : \text{Pol}(\text{Mat}_{mn})_q \rightarrow \mathbb{C}$ given by $l_\psi(f) = \int_{\mathbb{U}_q} f \psi d\nu$. Let \mathcal{T} be the weakest among the topologies in $\text{Pol}(\text{Mat}_{mn})_q$ in which all the linear functionals l_ψ , $\psi \in D(\mathbb{U})_q$, are continuous.

Consider the *finite dimensional* subspaces $\text{Pol}(\text{Mat}_{mn})_{q,i,j} = \mathbb{C}[\text{Mat}_{mn}]_{q,i} \cdot \mathbb{C}[\overline{\text{Mat}_{mn}}]_{q,j}$, $i \leq 0, j \geq 0$, of the vector space $\text{Pol}(\text{Mat}_{mn})_q$, together with such linear operators $p_{ij} : \text{Pol}(\text{Mat}_{mn})_q \rightarrow \text{Pol}(\text{Mat}_{mn})_{q,i,j}$ that $f = \sum_{i \geq 0, j \leq 0} p_{ij}(f)$ for all $f \in \text{Pol}(\text{Mat}_{mn})_q$. Let \mathcal{T}_1 be the weakest among the topologies in $\text{Pol}(\text{Mat}_{mn})_q$ in which all p_{ij} , $i \leq 0, j \geq 0$, are continuous.

Consider the *finite dimensional* orthogonal projections P_i in \mathcal{H} onto the homogeneous components \mathcal{H}_i , $i \in \mathbb{Z}_+$, and the linear operators $\Theta_{ij} : \text{Pol}(\text{Mat}_{mn})_q \rightarrow \text{Hom}(\mathcal{H}_j, \mathcal{H}_i)$, $f \mapsto P_i \Theta(f)|_{\mathcal{H}_j}$, $i, j \in \mathbb{Z}_+$.

Let \mathcal{T}_2 be the weakest among the topologies in which all the linear operators Θ_{ij} , $i, j \in \mathbb{Z}_+$, are continuous.

The completion of $\text{Pol}(\text{Mat}_{mn})_q$ with respect to \mathcal{T} will be denoted by $D(\mathbb{U})'_q$ and called the space of distributions in the quantum matrix ball. The pairing $\text{Pol}(\text{Mat}_{mn})_q \times D(\mathbb{U})_q \rightarrow \mathbb{C}$, $f \times \psi \mapsto \int_{\mathbb{U}_q} f\psi d\nu$, is extendable up to a pairing $D(\mathbb{U})'_q \times D(\mathbb{U})_q \rightarrow \mathbb{C}$, $f \times \psi \mapsto \int_{\mathbb{U}_q} f\psi d\nu$; this justifies the use of the term 'distribution'.

One can replace the topology \mathcal{T} in the definition of $D(\mathbb{U})'_q$ with either \mathcal{T}_1 or \mathcal{T}_2 , as it follows from

Proposition 4.1 *The topologies \mathcal{T} , \mathcal{T}_1 , \mathcal{T}_2 in $\text{Pol}(\text{Mat}_{mn})_q$ are equivalent.*

Proof. By [7, remark 8.6], $D(\mathbb{U})_q \xrightarrow[\Theta]{\sim} \text{End}(\mathcal{H})_f$, with $\text{End}(\mathcal{H})_f \simeq \mathcal{H} \otimes \mathcal{H}^*$. Thus, the equivalence of the topologies \mathcal{T} , \mathcal{T}_2 follows from (3.1), (3.6).

What remains is to prove the equivalence of \mathcal{T}_1 and \mathcal{T}_2 . Consider the linear span \mathcal{L}_1 of the images under the embedding $(\text{Pol}(\text{Mat}_{mn})_q)_{ij}^* \hookrightarrow (\text{Pol}(\text{Mat}_{mn})_q)^*$ and the linear span \mathcal{L}_2 of the images under the embedding $(\text{Hom}(\mathcal{H}_j, \mathcal{H}_i))^* \hookrightarrow (\text{Pol}(\text{Mat}_{mn})_q)^*$. It suffices to prove that $\mathcal{L}_1 = \mathcal{L}_2$. The inclusion $\mathcal{L}_1 \supseteq \mathcal{L}_2$ follows from $\text{Hom}(\mathcal{H}_j, \mathcal{H}_i)^* \subseteq \bigoplus_{k=0}^{\min(i,j)} \text{Pol}(\text{Mat}_{mn})_{q,i-k,-j+k}^*$. The converse inclusion follows from $\text{Pol}(\text{Mat}_{mn})_{q,i-j}^* \subseteq \bigoplus_{k=0}^{\min(i,j)} \text{Hom}(\mathcal{H}_j, \mathcal{H}_i)^*$. The latter inclusion is easily deducible from the previous one and [7, lemma 8.7]. ■

The equivalence of \mathcal{T} and \mathcal{T}_1 allows one to identify the topological vector space $D(\mathbb{U})'_q$ and the space of formal series $f = \sum_{i \geq 0, j \leq 0} f_{ij}$, $f_{ij} \in \text{Pol}(\text{Mat}_{mn})_{q,i,j}$ equipped with the topology of coefficientwise convergence. The structure of a covariant $\text{Pol}(\text{Mat}_{mn})_q$ -bimodule is transferred by a continuity from $\text{Pol}(\text{Mat}_{mn})_q$ onto the above space $D(\mathbb{U})'_q$ of formal series.

Let $\overline{\text{End}}(\mathcal{H}) = \prod_{i,j \geq 0} \text{Hom}(\mathcal{H}_j, \mathcal{H}_i)$ – the direct product in the category of vector spaces, i.e. the corresponding space of formal series. Evidently, $\text{End}(\mathcal{H})_f \hookrightarrow \text{End}(\mathcal{H}) \hookrightarrow \overline{\text{End}}(\mathcal{H})$.

Consider the embedding $i : D(\mathbb{U})_q \hookrightarrow D(\mathbb{U})'_q$ determined via the isomorphisms $D(\mathbb{U})_q \simeq \text{End}(\mathcal{H})_f$, $D(\mathbb{U})'_q \simeq \overline{\text{End}}(\mathcal{H})$. (The second isomorphism is a consequence of the equivalence of \mathcal{T} and \mathcal{T}_2 .)

Proposition 4.2 *The embedding of vector spaces $i : D(\mathbb{U})_q \hookrightarrow D(\mathbb{U})'_q$ is a morphism of covariant $\text{Pol}(\text{Mat}_{mn})_q$ -modules.*

Proof. By the construction, i is a morphism of $\text{Pol}(\text{Mat}_{mn})_q$ -bimodules. What remains is to prove that this is a morphism of $U_q\mathfrak{sl}_N$ -modules. Consider the element $f_0 \in D(\mathbb{U})'_q$ which is the image of $f_0 \in D(\mathbb{U})_q$ under the embedding i . It suffices to demonstrate that the relations from [7] which define the structure of $U_q\mathfrak{sl}_N$ -module in $D(\mathbb{U})_q$, are also valid in $D(\mathbb{U})'_q$:

$$\begin{aligned} H_n f_0 &= 0, & F_n f_0 &= \frac{q^{1/2}}{q^{-2} - 1} f_0(z_n^m)^*, & E_n f_0 &= -\frac{q^{1/2}}{1 - q^2} z_n^m f_0, \\ H_k f_0 &= F_k f_0 = E_k f_0 = 0 & \text{for } k &\neq n. \end{aligned}$$

It is easy to see that

$$\mathbb{C}f_0 = \{f \in D(\mathbb{U})'_q \mid (z_a^\alpha)^* f = f z_a^\alpha = 0, \quad \alpha = 1, \dots, m; a = 1, \dots, n\}.$$

It follows from the covariance of the $\text{Pol}(\text{Mat}_{mn})_q$ -bimodule $D(\mathbb{U})'_q$ that the subspace $\mathbb{C}f_0$ is a $U_q\mathfrak{sl}_n \times \mathfrak{gl}_m$ -submodule. Hence, $H_n f_0 = 0$, $H_k f_0 = F_k f_0 = E_k f_0 = 0$, $k \neq n$. Similarly,

$$\begin{aligned} \mathbb{C}z_n^m f_0 &= \{f \in D(\mathbb{U})'_q \mid H_0 f = 2f \text{ \& } F_j f = 0 \text{ for } j \neq n \\ &\quad \& f z_a^\alpha = 0 \text{ for } \alpha = 1, \dots, m; a = 1, \dots, n\}. \\ \mathbb{C}f_0(z_n^m)^* &= \{f \in D(\mathbb{U})'_q \mid H_0 f = -2f \text{ \& } E_j f = 0 \text{ for } j \neq n \\ &\quad \& (z_a^\alpha)^* f = 0 \text{ for } \alpha = 1, \dots, m; a = 1, \dots, n\}. \end{aligned}$$

Apply the covariance of the $\text{Pol}(\text{Mat}_{mn})_q$ -bimodule $D(\mathbb{U})'_q$ to get

$$F_n f_0 = \text{const}_1 f_0(z_n^m)^*, \quad E_n f_0 = \text{const}_2 z_n^m f_0.$$

What remains is to prove that $\text{const}_1 = -\frac{q^{1/2}}{q^{-2} - 1}$, $\text{const}_2 = -\frac{q^{1/2}}{1 - q^2}$.

The first constant is accessible from the relations

$$F_n(f_0 z_n^m) = F_n 0 = 0, \quad f_0(1 - (z_n^m)^* z_n^m) = q^2 f_0(1 - z_n^m(z_n^m)^*) = q^2 f_0,$$

and the second one follows from

$$E_n((z_n^m)^* f_0) = E_n 0 = 0, \quad (1 - (z_n^m)^* z_n^m) f_0 = q^2 f_0(1 - z_n^m(z_n^m)^*) f_0 = q^2 f_0.$$

(A detailed exposition of these calculations can be found in [7]). ■

We identify in the sequel finite functions $f \in D(\mathbb{U})_q$ with their images $i(f) \in D(\mathbb{U})'_q$ under the embedding i .

REMARK 4.3. By definition, the linear subspace $D(\mathbb{U})_q \xrightarrow[\ominus]{\sim} \text{End}(\mathcal{H})_f$ is dense in the topological vector space $D(\mathbb{U})'_q \xrightarrow[\ominus]{\sim} \overline{\text{End}(\mathcal{H})}$. The structure of the $D(\mathbb{U})_q$ -bimodule is extendable by a continuity from this dense linear subspace onto the entire space $D(\mathbb{U})'_q$. $D(\mathbb{U})_q$ is a covariant algebra, hence the $D(\mathbb{U})_q$ -bimodule we have obtained is also covariant.

To conclude, we prove the following

Proposition 4.4 *A distribution $f \in D(\mathbb{U})'_q$ is a finite function iff $f\mathbb{C}[\text{Mat}_{mn}]_{q,M} = \mathbb{C}[\overline{\text{Mat}_{mn}}]_{q,-M}f = 0$ for some $M \in \mathbb{N}$.*

Proof. Apply the fact that the linear map $\mathbb{C}[\text{Mat}_{mn}]_q \rightarrow \mathcal{H}$, $\psi \mapsto \psi f_0$, is one-to-one, and $\mathbb{C}[\text{Mat}_{mn}]_q$ is a domain [2]. For all $k \in \mathbb{Z}_+$, the linear span of $\Theta(z_a^\alpha) \mathcal{H}_k$, $\alpha = 1, \dots, m$, $a = 1, \dots, n$, coincides with \mathcal{H}_{k+1} , and the linear span of $\Theta((z_a^\alpha)^*) \mathcal{H}_k$, $\alpha = 1, \dots, m$, $a = 1, \dots, n$, coincides with \mathcal{H}_{k-1} (or is 0 in the case $k = 0$). what remains is to apply the isomorphism of vector spaces $D(\mathbb{U})_q \xrightarrow[\Theta]{} \text{End}(\mathcal{H})_f \simeq \bigoplus_{i \geq 0, j \geq 0} \mathcal{H}_i \otimes \mathcal{H}_j^* \simeq \bigoplus_{i \geq 0, j \geq 0} \mathcal{H}_i \otimes \mathcal{H}_j$. ■

5 Differential forms with finite coefficients

The results of this section are not used in producing an explicit formula for the Bergman kernel. However, it makes an independent interest and provides an essential addition to the results of [7]. In [6] a covariant algebra $\Omega(\text{Mat}_{mn})_q$ of differential forms with polynomial coefficients was considered (it was denoted there by $\Omega(\mathfrak{g}_{-1})_q$).

One can find in [7, section 4] a complete list of relations between the generators z_a^α , $(z_a^\alpha)^*$, dz_a^α , $d(z_a^\alpha)^*$, $a = 1, \dots, n$, $\alpha = 1, \dots, m$, of $\Omega(\text{Mat}_{mn})_q$. Consider the subalgebras $\Lambda_{mn} \subset \Omega(\text{Mat}_{mn})_q$, $\overline{\Lambda}_{mn} \subset \Omega(\text{Mat}_{mn})_q$, generated by $\{dz_a^\alpha\}$, $\{d(z_a^\alpha)^*\}$ respectively. They are q -analogues of algebras of differential forms with constant coefficients, and $\dim \Lambda_{mn} = \dim \overline{\Lambda}_{mn} = 2^{mn}$. There is a decomposition

$$\Omega(\text{Mat}_{mn})_q = \Lambda_{mn} \otimes \text{Pol}(\text{Mat}_{mn})_q \otimes \overline{\Lambda}_{mn}.$$

We are interested in considering the space $\Omega(\mathbb{U})_q \stackrel{\text{def}}{=} \Lambda_{mn} \otimes D(\mathbb{U})_q \otimes \overline{\Lambda}_{mn}$ of differential forms with finite coefficients and the space $\Omega_q \stackrel{\text{def}}{=} \Lambda_{mn} \otimes \text{Fun}(\mathbb{U})_q \otimes \overline{\Lambda}_{mn} = \Omega(\text{Mat}_{mn})_q + \Omega(\mathbb{U})_q$. We are going to equip Ω_q with a structure of covariant differential algebra and to describe it in terms of generators and relations.

Use the above topology in $D(\mathbb{U})'_q$ to introduce a topology in the vector space $\Omega'(\mathbb{U})_q = \Lambda_{mn} \otimes D(\mathbb{U})'_q \otimes \overline{\Lambda}_{mn} = \Lambda(\text{Mat}_{mn})_q \otimes_{\mathbb{C}[\text{Mat}_{mn}]_q} D(\mathbb{U})'_q \otimes_{\mathbb{C}[\overline{\text{Mat}}_{mn}]_q} \Lambda(\overline{\text{Mat}}_{mn})_q$ of differential forms whose coefficients are distributions. The differential d and the structure of covariant $\Omega(\text{Mat}_{mn})_q$ -bimodule are transferred by a continuity from $\Omega(\text{Mat}_{mn})_q$ onto $\Omega'(\mathbb{U})_q$. (In fact, all the commutation relations involving the differentials dz_a^α , $d(z_a^\alpha)^*$ are purely quadratic). Using proposition 4.4, it is easy to distinguish the differential forms with finite coefficients from $\Omega'(\mathbb{U})_q$, that is, to prove that

$$\Omega(\mathbb{U})_q = \{\omega \in \Omega'(\mathbb{U})_q \mid \exists M : \mathbb{C}[\overline{\text{Mat}}_{mn}]_{q,-M} \cdot \omega = \omega \cdot \mathbb{C}[\text{Mat}_{mn}]_{q,M} = 0\}.$$

This allows one to extend by a continuity the structure of covariant differential algebra from $\Omega(\text{Mat}_{mn})_q$ onto $\Omega_q = \Omega(\text{Mat}_{mn})_q + \Omega(\mathbb{U})_q$.

A complete list of commutation relations between the generators z_a^α , $(z_a^\alpha)^*$, f_0 , dz_a^α , $d(z_a^\alpha)^*$, $a = 1, \dots, n$; $\alpha = 1, \dots, m$, of Ω_q , includes (5.1) and the relations from [7]. (5.2) describes the action of the differential d onto f_0 .

Proposition 5.1 *For all $a = 1, \dots, n$; $\alpha = 1, \dots, m$, one has*

$$f_0 d(z_a^\alpha)^* = d(z_a^\alpha)^* f_0, \quad f_0 dz_a^\alpha = dz_a^\alpha f_0. \quad (5.1)$$

Proof. It suffices to consider the first relation. It follows from the invertibility of R-matrices involved into the commutation relations between $d(z_a^\alpha)^*$ and z_a^α , $(z_a^\alpha)^*$, $a = 1, \dots, n$; $\alpha = 1, \dots, m$, that

$$f_0 d(z_a^\alpha)^* = \sum_{b=1}^n \sum_{\beta=1}^m d(z_b^\beta)^* \psi_{\beta a}^{b\alpha}, \quad \psi_{\beta a}^{b\alpha} \in D'(\mathbb{U})_q.$$

Prove that $\psi_{\beta a}^{b\alpha} \cdot z_c^\gamma = 0$ for all $c = 1, \dots, n$; $\gamma = 1, \dots, m$. In fact, $f_0 \mathbb{C}[\text{Mat}_{mn}]_{q,1} = 0$. Hence,

$$0 = f_0 d(z_a^\alpha)^* z_c^\gamma = \sum_{b=1}^n \sum_{\beta=1}^m d(z_b^\beta)^* (\psi_{\beta a}^{b\alpha} \cdot z_c^\gamma)$$

(the first equality is due to the homogeneity of the commutation relations between $d(z_b^\beta)^*$ and z_c^γ , $a, c = 1, \dots, n$; $\alpha, \gamma = 1, \dots, m$). Thus, it follows from the latter equality and the definition of $\Omega'(\mathbb{U})_q$ that $\psi_{\beta a}^{b\alpha} \cdot z_c^\gamma = 0$, $c = 1, \dots, n$; $\gamma = 1, \dots, m$. \blacksquare

Lemma 5.2 *If $\psi \in D'(\mathbb{U})_q$ is such that $\psi \cdot z_c^\gamma = 0$, $c = 1, \dots, n$; $\gamma = 1, \dots, m$, then $\psi \in \bigtimes_{j=0}^{\infty} \mathbb{C}[\text{Mat}_{mn}]_{q,j} \cdot f_0 \subset D'(\mathbb{U})_q$.*

Proof. Show first that for any $\psi \in D'(\mathbb{U})_q$ one has $\psi y^N \xrightarrow{N \rightarrow \infty} \psi f_0$ in the topology of $D'(\mathbb{U})_q$. In fact, it suffices to demonstrate that $l_\varphi(\psi y^N) \xrightarrow{N \rightarrow \infty} l_\varphi(\psi f_0)$ for any $\varphi \in D'(\mathbb{U})_q$. Apply the decomposition $D(\mathbb{U})_q = \bigoplus_{\substack{j \geq 0 \\ k \leq 0}} D(\mathbb{U})_{q,j,k}$, with $D(\mathbb{U})_{q,j,k} = \mathbb{C}[\text{Mat}_{mn}]_{q,j} \cdot f_0 \cdot \mathbb{C}[\overline{\text{Mat}}_{mn}]_{q,k}$. One has: $\varphi = \sum_{j,k} \varphi_{jk}$, with $\varphi_{jk} \in D(\mathbb{U})_{q,j,k}$,

$$l_\varphi(\psi y^N) = \int_{\mathbb{U}_q} \psi y^N \varphi d\nu = \sum_{j,k} \int_{\mathbb{U}_q} \psi y^N \varphi_{jk} d\nu = \sum_{j,k} q^{2Nj} \int_{\mathbb{U}_q} \psi \varphi_{jk} d\nu.$$

On the other hand,

$$\lim_{N \rightarrow \infty} \sum_{j,k} q^{2Nj} \int_{\mathbb{U}_q} \psi \varphi_{jk} d\nu = \sum_k \int_{\mathbb{U}_q} \psi \varphi_{0k} d\nu = \sum_{j,k} \int_{\mathbb{U}_q} \psi f_0 \varphi_{jk} d\nu = \int_{\mathbb{U}_q} \psi f_0 \varphi d\nu = l_\varphi(\psi f_0).$$

Turn back to the proof of lemma 5.2. If $\psi \in D'(\mathbb{U})_q$ and $\psi \cdot z_c^\gamma = 0$ for all $c = 1, \dots, n$; $\gamma = 1, \dots, m$, then it follows from theorem 2.5 that $\psi y = \psi$, and hence $\psi = \psi y^N \xrightarrow{N \rightarrow \infty} \psi f_0$, so the statement of the lemma is proved. \blacksquare

We have demonstrated that $\psi \in \bigtimes_{j=0}^{\infty} \mathbb{C}[\text{Mat}_{mn}]_{q,j} \cdot f_0$ ($a, b = 1, \dots, n$; $\alpha, \beta = 1, \dots, m$).

Prove that $\psi_{\beta a}^{b\alpha}$ differs from f_0 only by a constant multiple. All $U_q \mathfrak{sl}_N$ -modules considered in [7] were equipped with a gradation determined by the element H_0 of the Cartan subalgebra:

$$H_0 \stackrel{\text{def}}{=} \frac{2}{m+n} \left(m \sum_{j=1}^{n-1} j H_j + n \sum_{j=1}^{m-1} j H_{N-j} + mn H_n \right).$$

An application of the operator q^{H_0} to $f_0 d(z_a^\alpha)^*$ and $d(z_b^\beta)^* \psi_{\beta a}^{b\alpha}$ yields $q^{H_0}(\psi_{\beta a}^{b\alpha}) = \psi_{\beta a}^{b\alpha}$. What remains is to remind that $\{f \in \bigtimes_{j=0}^{\infty} \mathbb{C}[\text{Mat}_{mn}]_{q,j} \cdot f_0 \mid q^{H_0} f = f\} = \mathbb{C} f_0$.

We have proved that the linear span of $\{f_0 \cdot d(z_a^\alpha)^*\}_{a=1,\dots,n; \alpha=1,\dots,m}$ coincides with the linear span of $\{d(z_a^\alpha)^* f_0\}_{b=1,\dots,n; \beta=1,\dots,m}$. This vector space is a simple $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m$ -module, and the linear map $f_0 \cdot d(z_a^\alpha)^* \mapsto d(z_a^\alpha)^* f_0$ is the unique up to a constant multiple endomorphism of this $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m$ -module. Hence, $f_0 \cdot d(z_a^\alpha)^* = C \cdot d(z_a^\alpha)^* f_0$ for some $C \in \mathbb{C}$. On the other hand, $f_0^* = f_0$, and so $C \cdot d(z_a^\alpha)^* f_0 = f_0 \cdot d(z_a^\alpha)^* = f_0^2 d(z_a^\alpha)^* = C^2 \cdot d(z_a^\alpha)^* f_0^2 = C^2 \cdot d(z_a^\alpha)^* f_0$. Therefore, $(1 - C)C \cdot d(z_a^\alpha)^* f_0 = 0$. On the other hand, $f_0 \cdot d(z_a^\alpha)^* = C \cdot d(z_a^\alpha)^* f_0 \neq 0$, and thus we get $C = 1$. \blacksquare

Proposition 5.3

$$df_0 = -\frac{1}{1-q^2} \sum_{a=1}^n \sum_{\alpha=1}^m (dz_a^\alpha f_0 (z_a^\alpha)^* + z_a^\alpha f_0 (dz_a^\alpha)^*). \quad (5.2)$$

Proof. It follows from $f_0^* = f_0$ that $df_0 = \bar{\partial} f_0 + \partial f_0 = \bar{\partial} f_0 + (\bar{\partial} f_0)^* = \bar{\partial} f_0 + (\bar{\partial} f_0)^*$. Hence, it suffices to prove the relation

$$\bar{\partial} f_0 = \omega_0, \quad \omega_0 = -\frac{1}{1-q^2} \sum_{a=1}^n \sum_{\alpha=1}^m z_a^\alpha f_0 d(z_a^\alpha)^*.$$

Let $\Omega(\mathbb{U})_q^{(0,1)}$ be the space of (0,1)-forms with finite coefficients in the quantum ball. Remind that all the $U_q \mathfrak{sl}_N$ -modules in our consideration are equipped with a \mathbb{Z} -grading defined by the element H_0 [7]. We are about to prove that 1-forms $\bar{\partial} f_0$ and ω_0 are solutions of the following system of equations:

$$(1-y)\omega = \bar{\partial} y \cdot f_0, \quad H_0 \omega = 0 \quad (5.3)$$

and to elaborate the uniqueness of a solution of this system in the space $\Omega(\mathbb{U})_q^{(0,1)}$.

We start with proving the uniqueness.

Lemma 5.4 *If $\omega \in \Omega(\mathbb{U})_q^{(0,1)}$ and $(1-y)\omega = H_0 \omega = 0$, then $\omega = 0$.*

Proof. Apply the decomposition $D(\mathbb{U})_q = \bigoplus_{j \geq 0, k \leq 0} D(\mathbb{U})_{q,j,k}$, with $D(\mathbb{U})_{q,j,k} = \mathbb{C}[\text{Mat}_{mn}]_{q,j} f_0 \mathbb{C}[\overline{\text{Mat}}_{mn}]_{q,k}$. One has $\omega = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \omega_{jk}$, where $\omega_{jk} = \sum_{a=0}^n \sum_{\alpha=0}^m f_{\alpha j k}^a d(z_a^\alpha)^*$, $f_{\alpha j k} \in D(\mathbb{U})_{q,j,-k}$. The statement of the lemma now follows from the relations $H_0 \omega_{jk} = 2(j-k-1)\omega_{jk}$ and $(1-y)\omega_{jk} = (1-q^{2j})\omega_{jk}$. (The latter relation is deducible from $y f_0 = f_0$ and $y z_a^\alpha = q^2 z_a^\alpha y$; $a = 1, \dots, n$; $\alpha = 1, \dots, m$). \blacksquare

Turn back to the proof of proposition 5.2. What remains is to verify that the 1-forms $\bar{\partial} f_0$ and ω_0 are solutions of the equation system 5.3.

The relations $H_0(\bar{\partial} f_0) = 0$, $H_0 \omega_0 = 0$ follow from $H_0 f_0 = 0$, $H_0 z_a^\alpha = 2z_a^\alpha$, $H_0(dz_a^\alpha)^* = -2(dz_a^\alpha)^*$, $\alpha = 1, \dots, m$, $a = 1, \dots, n$, together with the fact that $\bar{\partial}$ is a morphism of $U_q \mathfrak{sl}_N$ -modules.

To prove the relation $(1 - y)\bar{\partial}f_0 = \bar{\partial}y \cdot f_0$ it is sufficient to apply $\bar{\partial}$ to both sides of the equality $y \cdot f_0 = f_0$.

The relation $(1 - y) \cdot \omega_0 = \bar{\partial}y \cdot f_0$ follows from

$$\bar{\partial}y \cdot f_0 = - \sum_{a=1}^n \sum_{\alpha=1}^m z_a^\alpha f_0 d(z_a^\alpha)^*. \quad (5.4)$$

On the other hand, (5.4) is deducible from the explicit formula for the element y obtained in section 2:

$$y = \sum_{k=0}^{\infty} (-1)^k y_k, \quad y_0 = 1, \quad y_1 = \sum_{a=1}^n \sum_{\alpha=1}^m z_a^\alpha (z_a^\alpha)^*,$$

$$y_k = \sum_{\{J' | \text{card}(J')=k\}} \sum_{\{J'' | \text{card}(J'')=k\}} z^{\wedge k}_{J''} \left(z^{\wedge k}_{J'} \right)^*.$$

In fact, $d(z_a^\alpha)^* f_0 = f_0 d(z_a^\alpha)^*$; $(z_a^\alpha)^* f_0 = 0$ for all $a = 1, \dots, n$; $\alpha = 1, \dots, m$, and hence $\bar{\partial}y_k \cdot f_0 = 0$ for all $k \geq 2$. \blacksquare

6 The linear map $\overset{\circ}{P}_\lambda$

We assume in what follows $\lambda > N - 1$.

Proposition 6.1 *The linear subspace $D(\mathbb{U})_q$ is dense in the Hilbert space $L^2(d\nu_\lambda)_q$, and the embedding $D(\mathbb{U})_q \hookrightarrow D(\mathbb{U})'_q$ extends by a continuity up to an embedding $L^2(d\nu_\lambda)_q \hookrightarrow D(\mathbb{U})'_q$.*

Proof. It suffices to apply the relations (3.1), (3.5), and the isomorphisms

$$D(\mathbb{U})_q \xrightarrow{\sim} \text{End}(\mathcal{H})_f, \quad D(\mathbb{U})'_q \xrightarrow{\sim} \overline{\text{End}}(\mathcal{H}) \quad (\text{see section 4}) \quad \blacksquare$$

We identify in the sequel the Hilbert space $L^2(d\nu_\lambda)_q$ and its image under the embedding into the space of distributions $D(\mathbb{U})'_q$.

Consider the orthogonal projection P_λ in the Hilbert space $L^2(d\nu_\lambda)_q$ onto the Hardy-Bergman subspace $L^2_a(d\nu_\lambda)_q$ introduced in section 3. A principal subject of the research in the remainder of this work will be the linear map $\overset{\circ}{P}_\lambda: D(\mathbb{U})_q \rightarrow D(\mathbb{U})'_q$ given by a restriction of P_λ onto the dense in $L^2(d\nu_\lambda)_q$ linear subspace $D(\mathbb{U})_q$.¹

This section presents a construction of such a representation π_λ of $U_q \mathfrak{sl}_N$ in $D(\mathbb{U})'_q$ that $\pi_\lambda(a)D(\mathbb{U})_q \subset D(\mathbb{U})_q$ for all $a \in U_q \mathfrak{sl}_N$, and

$$\pi_\lambda(a) \overset{\circ}{P}_\lambda = \overset{\circ}{P}_\lambda \pi_\lambda(a), \quad a \in U_q \mathfrak{sl}_N. \quad (6.1)$$

The results of section 7 will imply that P_λ is uniquely determined by (6.1) and its value on f_0 .

¹It will be proved in the sequel that $P_\lambda \text{Fun}(\mathbb{U})_q \subset \mathbb{C}[\text{Mat}_{mn}]_q$

Proposition 6.2 *There exists a unique representation π_λ of $U_q\mathfrak{sl}_N$ in $D(\mathbb{U})'_q$ such that for all $f \in D(\mathbb{U})'_q$*

$$\pi_\lambda(E_j) : f \mapsto \begin{cases} E_j f, & j \neq n, \\ E_n f - q^{1/2} \frac{1 - q^{2\lambda}}{1 - q^2} (K_n f) z_n^m, & j = n. \end{cases} \quad (6.2)$$

$$\pi_\lambda(F_j) : f \mapsto \begin{cases} F_j f, & j \neq n, \\ q^{-\lambda} F_n f, & j = n. \end{cases} \quad (6.3)$$

$$\pi_\lambda(K_j^{\pm 1}) : f \mapsto \begin{cases} K_j^{\pm 1} f, & j \neq n, \\ q^{\pm \lambda} K_n^{\pm 1} f, & j = n. \end{cases} \quad (6.4)$$

Proof. The uniqueness of π_λ is obvious. While proving the existence of this representation, it suffices to replace the topological vector space $D(\mathbb{U})'_q$ by its dense subspace $\text{Pol}(\text{Mat}_{mn})_q$, and to consider the special case $\lambda \in \{N, N+1, N+2, \dots\}$. (In fact, the problem is to prove equalities in which both sides are in $\mathbb{C}[q^\lambda, q^{-\lambda}]$. So, what remains is to observe that two polynomials which coincide on the set $\{q^N, q^{N+1}, q^{N+2}, \dots\}$ are identically the same.) Consider the $U_q\mathfrak{sl}_N$ -module $\text{Pol}(\tilde{X})_{q,x}$ and the associated representation π of $U_q\mathfrak{sl}_N$. Remind the notation $t = t_{\{1,2,\dots,m\}\{n+1,n+2,\dots,N\}}^{\wedge m}$. The existence of π_λ follows from the following

Lemma 6.3 *Let $\lambda \in \{N, N+1, N+2, \dots\}$ and \mathcal{J}_λ be the linear map $\mathcal{J}_\lambda : \text{Pol}(\text{Mat}_{mn})_q \rightarrow \text{Pol}(\tilde{X})_{q,x}$; $\mathcal{J}_\lambda : f \mapsto (\mathcal{J}f)t^{-\lambda}$. Then for all $j = 1, \dots, N-1$ one has $\pi_\lambda(E_j) = \mathcal{J}_\lambda^{-1}\pi(E_j)\mathcal{J}_\lambda$, $\pi_\lambda(F_j) = \mathcal{J}_\lambda^{-1}\pi(F_j)\mathcal{J}_\lambda$, $\pi_\lambda(K_j^{\pm 1}) = \mathcal{J}_\lambda^{-1}\pi(K_j^{\pm 1})\mathcal{J}_\lambda$.*

Proof. It follows from the covariance of the algebra $D(\mathbb{U})'_q$ that

$$\begin{aligned} E_j(ft^{-\lambda}) &= (E_j f)(t^{-\lambda}) + (K_j f)(E_j(t^{-\lambda})), \\ F_j(ft^{-\lambda}) &= (F_j f)(K_j^{-1}(t^{-\lambda})) + f \cdot (F_j(t^{-\lambda})), \\ K_j^{\pm 1}(ft^{-\lambda}) &= (K_j^{\pm 1} f)(K_j^{\pm 1}(t^{-\lambda})), \end{aligned}$$

so it suffices to prove the relations

$$\begin{aligned} F_j(t^{-\mu}) &= 0, \quad K_j^{\pm 1}(t^{-\mu}) = \begin{cases} q^{\pm \mu} t^{-\mu}, & j = n \\ t^{-\mu}, & j \neq n. \end{cases} \\ E_j(t^{-\mu}) &= \begin{cases} -q^{1/2} \frac{1 - q^{2\mu}}{1 - q^2} \left(t^{-1} \cdot t_{\{1,2,\dots,m\}\{n,n+2,\dots,N\}}^{\wedge m} \right) t^{-\mu}, & j = n, \\ 0, & j \neq n. \end{cases} \end{aligned}$$

for all $\mu \in -\mathbb{Z}_+$. These are easily deducible via an application of the covariance of $\text{Pol}(\tilde{X})_{q,x}$ and the relations

$$F_j t = 0, \quad K_j^{\pm 1} t = \begin{cases} q^{\mp 1} t, & j = n \\ t, & j \neq n. \end{cases},$$

$$E_j t = \begin{cases} q^{-1/2} \cdot t^{\wedge m}_{\{1,2,\dots,m\}\{n,n+2,\dots,N\}}, & j = n \\ 0, & j \neq n. \end{cases}$$

(For example,

$$\begin{aligned} E_n(t^k) &= \sum_{j=0}^{k-1} (K_n t)^j (E_n t) t^{k-j-1} = q^{-1/2} \sum_{j=0}^{k-1} (q^{-1} t)^j t^{\wedge m}_{\{1,2,\dots,m\}\{n,n+2,\dots,N\}} t^{k-j-1} = \\ &= q^{-3/2} \left(\sum_{j=0}^{k-1} q^{-2j} \right) (t^{-1} t^{\wedge m}_{\{1,2,\dots,m\}\{n,n+2,\dots,N\}}) t^k. \end{aligned} \quad \blacksquare$$

Remind the notion of an invariant scalar product in a representation space of a Hopf \ast -algebra A . Let S be the antipode and ε the counit of this Hopf algebra.

Consider an A -module V . The antimodule \overline{V} is defined to be V as an Abelian group, while the multiplication by complex numbers and A -action in \overline{V} are given by

$$(\lambda, v) \mapsto \overline{\lambda} v, \quad (a, v) \mapsto (S(a))^* v, \quad \lambda \in \mathbb{C}, \quad a \in A, \quad v \in \overline{V}.$$

Let V_1, V_2 be two A -modules. A sesquilinear form $V_1 \times V_2 \rightarrow \mathbb{C}$ is called invariant if the associated linear functional $\eta : \overline{V_2} \otimes V_1 \rightarrow \mathbb{C}$ is a morphism of A -modules:

$$\eta(av) = \varepsilon(a)\eta(v), \quad a \in A, \quad v \in \overline{V_2} \otimes V_1.$$

Consider the representation π_λ and its subrepresentation in $D(\mathbb{U})_q$. Let $D(\mathbb{U})_{q,\lambda}$ be the associated $U_q \mathfrak{su}_{n,m}$ -module. (The $U_q \mathfrak{su}_{n,m}$ -modules $\mathbb{C}[\overline{\text{Mat}}_{mn}]_{q,\lambda}$, $\text{Pol}(\overline{\text{Mat}}_{mn})_{q,\lambda}$, and $\text{Fun}(\mathbb{U})_{q,\lambda} = \text{Pol}(\overline{\text{Mat}}_{mn})_{q,\lambda} + D(\mathbb{U})_{q,\lambda}$ are defined in a similar way.)

Proposition 6.4 *For all $\lambda > N-1$, the scalar product $D(\mathbb{U})_{q,\lambda} \times D(\mathbb{U})_{q,\lambda} \rightarrow \mathbb{C}$, $f_1 \times f_2 \mapsto \int_{\mathbb{U}_q} f_2^* \cdot f_1 y^\lambda d\nu$ is positive and $U_q \mathfrak{su}_{n,m}$ -invariant.*

Proof. The positivity was demonstrated before (in section 3). The same argument as in the proof of proposition 6.2 allows one to reduce matters to the special case $\lambda \in \{N, N+1, N+2, \dots\}$.

In this special case one has a well defined operator $\mathcal{I}_\lambda : D(\mathbb{U})_{q,\lambda} \rightarrow D(\tilde{X})_q$, $\mathcal{I}_\lambda : f \mapsto (\mathcal{I}f)t^{-\lambda}$. It follows from lemma 6.3 that this linear map is a morphism of $U_q \mathfrak{su}_{n,m}$ -modules.

One can find in [7] a construction of invariant integral on a quantum principal homogeneous space. It is easy to deduce from that construction that $\int_{\tilde{X}_q} f t^* d\nu = \int_{\tilde{X}_q} t^* f d\nu$ for

all $f \in D(\tilde{X})_q$ since the operator $\Pi(tt^*)$ introduced in the paper alluded above commutes with $\Gamma(e^{h\hat{\rho}})$ involved in (3.1). Hence

$$\begin{aligned} \int_{\mathbb{U}_q} f_2^* f_1 y^\lambda d\nu &= \int_{\tilde{X}_q} (\mathcal{I}f_2)^* (\mathcal{I}f_1) (tt^*)^{-\lambda} d\nu = \int_{\tilde{X}_q} (\mathcal{I}f_2)^* (\mathcal{I}f_1) t^{-\lambda} (t^*)^{-\lambda} d\nu = \\ &= \int_{\tilde{X}_q} (t^*)^{-\lambda} (\mathcal{I}f_2)^* (\mathcal{I}f_1) t^{-\lambda} d\nu = \int_{\tilde{X}_q} (\mathcal{I}_\lambda f_2)^* (\mathcal{I}_\lambda f_1) d\nu. \end{aligned}$$

Thus, the invariance of scalar product in $D(\mathbb{U})_{q,\lambda}$ follows from the invariance of the scalar product $D(\tilde{X})_q \times D(\tilde{X})_q \rightarrow \mathbb{C}$, $f_1 \times f_2 \mapsto \int_{\tilde{X}_q} f_2^* f_1 d\nu$, while the latter statement follows from the invariance of the integral on the quantum principal homogeneous space.

Corollary 6.5 *For all $\lambda > N-1$, the scalar product $\text{Fun}(\mathbb{U})_{q,\lambda} \times \text{Fun}(\mathbb{U})_{q,\lambda} \rightarrow \mathbb{C}$, $f_1 \times f_2 \mapsto (f_1, f_2)_\lambda \stackrel{\text{def}}{=} C(\lambda) \int_{\mathbb{U}_q} f_2^* f_1 d\nu_\lambda$ is $U_q \mathfrak{su}_{n,m}$ -invariant (the constant $C(\lambda)$ is determined by (3.7)).*

Proof. Let $j \in \mathbb{Z}_+$ and $\chi_j \in D(\mathbb{U})_q$ be such a finite function that the operator $\Theta(\chi_j)$ in $\mathcal{H} = \bigoplus_{k=0}^\infty \mathcal{H}_k$ is the projection onto the space $\bigoplus_{k=0}^j \mathcal{H}_k$ along the subspace $\bigoplus_{k=j+1}^\infty \mathcal{H}_k$. (The existence and uniqueness of such χ is due to the isomorphism $\Theta : D(\mathbb{U})_q \rightarrow \text{End}(\mathcal{H})_f$). The invariance of the scalar product $(f_1, f_2)_\lambda$ in $D(\mathbb{U})_{q,\lambda}$ implies the invariance of the associated scalar product in $\text{Fun}(\mathbb{U})_{q,\lambda}$ since for all $f_1, f_2 \in \text{Fun}(\mathbb{U})_{q,\lambda}$, $a_1, a_2 \in U_q \mathfrak{su}_{n,m}$

$$(\pi_\lambda(a_1)f_1, \pi_\lambda(a_2)f_2)_\lambda = \lim_{m_1, m_2 \rightarrow \infty} (\pi_\lambda(a_1)(\chi_{m_1} f_1 \chi_{m_2}), \pi_\lambda(a_2)(\chi_{m_2} f_2 \chi_{m_1}))_\lambda.$$

While proving the latter equality, one should use the description of $U_q \mathfrak{sl}_N$ -action in $D(\mathbb{U})_q$ from [7, sections 7, 8]. ■

Proposition 6.6 $P_\lambda \text{Fun}(\mathbb{U})_{q,\lambda} \subset \mathbb{C}[\text{Mat}_{mn}]_{q,\lambda}$, and the associated linear map $P_\lambda : \text{Fun}(\mathbb{U})_{q,\lambda} \rightarrow \mathbb{C}[\text{Mat}_{mn}]_{q,\lambda}$ is a morphism of $U_q \mathfrak{su}_{n,m}$ -modules for all $\lambda > N-1$.

Proof. Each $f \in \text{Fun}(\mathbb{U})_{q,\lambda}$ is orthogonal to all but finitely many of homogeneous components of the graded vector space $\mathbb{C}[\text{Mat}_{mn}]_{q,\lambda}$. Hence $P_\lambda f \in \mathbb{C}[\text{Mat}_{mn}]_{q,\lambda}$. Since $P_\lambda^2 = P_\lambda$, it suffices to prove that $\mathbb{C}[\text{Mat}_{mn}]_{q,\lambda}$ and its orthogonal complement in $\text{Fun}(\mathbb{U})_{q,\lambda}$ are $U_q \mathfrak{su}_{n,m}$ -submodules of the $U_q \mathfrak{su}_{n,m}$ -module $\text{Fun}(\mathbb{U})_{q,\lambda}$. For the first subspace this follows from the definition of π_λ , and for the second one this fact is due to corollary 6.5 (by the invariance of the scalar product we have:

$$(\pi_\lambda(a)f_1, f_2)_\lambda = (f_1, \pi_\lambda(a^*)f_2)_\lambda, \quad a \in U_q \mathfrak{su}_{n,m}, \quad f_1, f_2 \in \text{Fun}(\mathbb{U})_{q,\lambda}$$

(cf. [8]). ■

Corollary 6.7 $P_\lambda D(\mathbb{U})_{q,\lambda} \subset \mathbb{C}[\text{Mat}_{mn}]_{q,\lambda}$, and the operator $\overset{\circ}{P}_\lambda : D(\mathbb{U})_{q,\lambda} \rightarrow \mathbb{C}[\text{Mat}_{mn}]_{q,\lambda}$ is a morphism of $U_q \mathfrak{su}_{n,m}$ -modules.

7 The element f_0

Consider the subalgebras $U_q \mathfrak{N}_\pm \subset U_q \mathfrak{sl}_N$ generated by $\{E_j\}_{j=1, \dots, N-1}$ and $\{F_j\}_{j=1, \dots, N-1}$, respectively.

Lemma 7.1 f_0 generates the $U_q\mathfrak{N}_+$ -module $\mathbb{C}[\text{Mat}_{mn}]_q f_0$ and the $U_q\mathfrak{N}_-$ -module $f_0 \mathbb{C}[\overline{\text{Mat}}_{mn}]_q$.

Proof. It suffices to prove the first statement. One can find in [6, 7] a description of the generalized Verma module $V_+(0)$ over $U_q\mathfrak{sl}_N$ with a single generator $v_+(0) \in V_+(0)$. It is sufficient to demonstrate that the map $v_+(0) \mapsto f_0$ admits an extension up to an isomorphism of the graded $U_q\mathfrak{N}_+$ -modules $V_+(0) \xrightarrow{\sim} \mathbb{C}[\text{Mat}_{mn}]_q f_0$. On the other hand, $\mathbb{C}[\text{Mat}_{mn}]_q f_0$ is a dual graded $U_q\mathfrak{N}_+$ -module with respect to $\mathbb{C}[\overline{\text{Mat}}_{mn}]_q$, due to the invariance and nondegeneracy of the bilinear form $\mathbb{C}[\text{Mat}_{mn}]_q f_0 \times \mathbb{C}[\overline{\text{Mat}}_{mn}]_q \rightarrow \mathbb{C}; f_1 \times f_2 \mapsto \int_{\mathbb{U}_q} f_1 f_2 d\nu$ (nondegeneracy follows from [7, lemma 8.4]). Furthermore, it was shown in [6, 7] that $V_+(0) \simeq (\mathbb{C}[\overline{\text{Mat}}_{mn}]_q)^*$ in the category of $U_q\mathfrak{sl}_N$ -modules. What remains is to refer to the coincidence of the kernels of the linear functionals associated to f_0 and $v_+(0)$ under the above isomorphisms. (These kernels are just $\bigoplus_{j=1}^{\infty} \mathbb{C}[\overline{\text{Mat}}_{mn}]_{q,-j}$). ■

Proposition 7.2 $U_q\mathfrak{sl}_N f_0 = D(\mathbb{U})_q$.

Proof. Consider an ordered set $\{j_1, j_2, \dots, j_r\}$, $r \in \mathbb{Z}_+$, formed by the elements of the set $\{1, 2, \dots, N-1\}$. It suffices to prove that $(E_{j_1} E_{j_2} \dots E_{j_r} f_0) \mathbb{C}[\overline{\text{Mat}}_{mn}]_q \subset U_q\mathfrak{sl}_N f_0$, since the linear span of $E_{j_1} E_{j_2} \dots E_{j_r} f_0$ coincides with $\mathbb{C}[\text{Mat}_{mn}]_q f_0$ by a virtue of lemma 7.1. We proceed by induction in r . In the case $r = 0$ our statement follows from lemma 7.1. The induction passage from $r-1$ to r could be easily done via an application of

$$(E_j(f_+ f_0))f_- = E_j(f_+ f_0 f_-) - K_j(f_+ f_0)(E_j f_-). \quad (7.1)$$

In fact, set up $j = j_1$, $f_+ f_0 = E_{j_2} E_{j_3} \dots E_{j_r} f_0$, $f_- \in \mathbb{C}[\overline{\text{Mat}}_{mn}]_q$. By the induction hypothesis one has

$$f_+ f_0 f_- \in (E_{j_2} E_{j_3} \dots E_{j_r} f_0) \mathbb{C}[\overline{\text{Mat}}_{mn}]_q \subset U_q\mathfrak{sl}_N f_0,$$

$$K_{j_1}(f_+ f_0)(E_{j_1} f_-) \in (E_{j_2} E_{j_3} \dots E_{j_r} f_0) \mathbb{C}[\overline{\text{Mat}}_{mn}]_q \subset U_q\mathfrak{sl}_N f_0.$$

Hence $(E_{j_1} E_{j_2} \dots E_{j_r} f_0) f_- = (E_{j_1}(f_+ f_0)) f_- = E_{j_1}(f_+ f_0 f_-) - K_{j_1}(f_+ f_0) E_{j_1} f_- \subset U_q\mathfrak{sl}_N f_0$. ■

The relations (6.2) – (6.4) allow one to generalize the statements of lemma 7.1 and proposition 7.2.

Lemma 7.3 $\{\pi_\lambda(\xi) f_0 \mid \xi \in U_q\mathfrak{N}_+\} = \mathbb{C}[\text{Mat}_{mn}]_q f_0$,
 $\{\pi_\lambda(\xi) f_0 \mid \xi \in U_q\mathfrak{N}_-\} = f_0 \mathbb{C}[\overline{\text{Mat}}_{mn}]_q$.

Proof. The first statement follows from lemma 7.1 since the action of the operators $\pi_\lambda(\xi)$, $\xi \in U_q\mathfrak{N}_+$, on the subspace $\mathbb{C}[\text{Mat}_{mn}]_q f_0$ is independent of λ . The second statement reduces to lemma 7.1 via replacement of the generator $F_n \mapsto q^{-\lambda} F_n$. ■

Proposition 7.4 $\{\pi_\lambda(\xi) f_0 \mid \xi \in U_q\mathfrak{sl}_N\} = D(\mathbb{U})_q$

Proof. Repeat the proof of proposition 7.2 with the reference to lemma 7.1 being replaced by that to lemma 7.3. The statement $(E_{j_1}E_{j_2}\dots E_{j_r}f_0)\mathbb{C}[\overline{\text{Mat}}_{mn}]_q \subset \{\pi_\lambda(\xi)f_0 \mid \xi \in U_q\mathfrak{sl}_N\}$ is proved by induction in r as before. The only difference is that the first term in the right hand side of (7.1) should be replaced in the case $j = n$ by $\pi_\lambda(E_n)(f_+f_0f_-) + q^{1/2}\frac{1-q^{2\lambda}}{1-q^2}K_n(f_+f_0f_-)z_n^m$. The appearance of the term $\text{const}(f_+, \lambda)f_+f_0K_n(f_-)z_n^m$ does not require introducing any essential changes to the induction process in question since $K_n(f_-) \in \mathbb{C}[\overline{\text{Mat}}_{mn}]_q$, $f_+f_0K_n(f_-)z_n^m \in f_+f_0\mathbb{C}[\overline{\text{Mat}}_{mn}]_q$. ■

REMARK 7.5. It follows from proposition 7.2 that the invariant integral $D(\mathbb{U})_q \rightarrow \mathbb{C}$ on the quantum matrix ball is unique up to a constant multiple.

8 The integral operators \widetilde{K}_l

This section contains a construction of a family of integral operators $\widetilde{K}_l : D(\mathbb{U})_q \rightarrow D(\mathbb{U})'_q$ which commute with the operators of the representation π_l .

Let $\mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}}$ be the graded algebra derived from $\mathbb{C}[\text{Mat}_{mn}]_q$ via replacing its multiplication law with the opposite one. The term 'algebra of kernels' will stand for a completion of the bigraded algebra $\mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}} \otimes \mathbb{C}[\overline{\text{Mat}}_{mn}]_q$, that is, the algebra of formal series of the form $K = \sum_{i,j=0}^{\infty} K^{(i,j)}$, $K^{(i,j)} \in \mathbb{C}[\text{Mat}_{mn}]_{q,i}^{\text{op}} \otimes \mathbb{C}[\overline{\text{Mat}}_{mn}]_{q,-j}$, with the topology of coefficientwise convergence (the topology of direct product). This algebra is denoted by $\mathbb{C}[[\text{Mat}_{mn} \times \overline{\text{Mat}}_{mn}]]_q$.

To begin with, we construct the kernels K_l of integral operators \widetilde{K}_l in the special case $l \in -\mathbb{N}$. A passage to the general case is to be performed later on via an 'analytic continuation' with respect to the parameter l in $\mathbb{C}[[\text{Mat}_{mn} \times \overline{\text{Mat}}_{mn}]]_q$ (cf. [8]).

It follows from the definition of the involutions $*$, \star that

$$(t_{\{1,2,\dots,m\}J}^{\wedge m})^* = (-1)^{(\text{card}\{1,2,\dots,n\} \cap J)} (t_{\{1,2,\dots,m\}J}^{\wedge m})^\star,$$

where $(t_{\{1,2,\dots,m\}J}^{\wedge m})^\star = (-q)^{l(J)} t_{\{m+1,\dots,N\}J'}^{\wedge n}$, $J' = \{1, \dots, N\} \setminus J$, $l(J) = \text{card}\{(a,b) \mid a > b \text{ \& } a \in J \text{ \& } b \in J'\}$. Apply these relations and the $U_q\mathfrak{sl}_N$ -invariance of the element $\sum_{s \in S_N} (-q)^{l(s)} t_{1s(1)} t_{2s(2)} \dots t_{ms(m)} \otimes t_{m+1s(m+1)} t_{m+2s(m+2)} \dots t_{Ns(N)} \in \mathbb{C}[SL_N]_q \otimes \mathbb{C}[SL_N]_q$ to obtain

Lemma 8.1 *The 'kernel'*

$$L = \sum_{\text{card}(J)=m, J \subset \{1,2,\dots,N\}} (-1)^{(\text{card}\{1,2,\dots,n\} \cap J)} t_{\{1,2,\dots,m\}J}^{\wedge m} \otimes (t_{\{1,2,\dots,m\}J}^{\wedge m})^* \quad (8.1)$$

is a $U_q\mathfrak{sl}_N$ -invariant of the $U_q\mathfrak{sl}_N$ -module $\text{Pol}(\widetilde{X})_q \otimes \text{Pol}(\widetilde{X})_q$. (That is, $aL = \varepsilon(a)L$ for all $a \in U_q\mathfrak{sl}_N$.)

Consider the algebra $\text{Pol}(\widetilde{X})_{q,x}^{\text{op}}$ which is coming from $\text{Pol}(\widetilde{X})_{q,x}$ via replacing its multiplication law with the opposite one.

REMARK 8.2. An application of proposition 2.4 allows one to prove that in the algebra $\text{Pol}(\tilde{X})_q^{\text{op}} \otimes \text{Pol}(\tilde{X})_q$

$$L = \mathcal{J} \otimes \mathcal{J} \left(1 + \sum_{k=1}^m (-1)^k \chi_k \right) t \otimes t^*,$$

with $\chi_k \in \mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}} \otimes \mathbb{C}[\text{Mat}_{mn}]_q \subset \mathbb{C}[[\text{Mat}_{mn} \times \overline{\text{Mat}}_{mn}]]_q$ being the kernels given by

$$\chi_k = \sum_{\substack{J' \subset \{1, \dots, m\} \\ \text{card}(J')=k}} \sum_{\substack{J'' \subset \{1, \dots, n\} \\ \text{card}(J'')=k}} z^{\wedge k J'}_{J''} \otimes \left(z^{\wedge k J'}_{J''} \right)^*. \quad (8.2)$$

It was shown in [9, section 2] that a product of any two $U_q \mathfrak{sl}_N$ -invariants of $\text{Pol}(\tilde{X})_q^{\text{op}} \otimes \text{Pol}(\tilde{X})_q$ is again a $U_q \mathfrak{sl}_N$ -invariant. Hence the following generalization of lemma 8.1.

Lemma 8.3 *All the powers L^j , $j \in \mathbb{N}$, of $L \in \text{Pol}(\tilde{X})_q^{\text{op}} \otimes \text{Pol}(\tilde{X})_q$, are $U_q \mathfrak{sl}_N$ -invariants.*

Let $l \in -\mathbb{N}$. Define the kernel K_l by

$$K_l = \left(1 + \sum_{k=1}^m (-q^{2l})^k \chi_k \right) \cdot \left(1 + \sum_{k=1}^m (-q^{2(l+1)})^k \chi_k \right) \cdots \left(1 + \sum_{k=1}^m (-q^{-2})^k \chi_k \right). \quad (8.3)$$

Corollary 8.4 *For all $l \in -\mathbb{N}$ the element $(t \otimes t^*)^{-l} \cdot \mathcal{J} \otimes \mathcal{J}(K_l)$ of $\text{Pol}(\tilde{X})_q^{\text{op}} \otimes \text{Pol}(\tilde{X})_q$ is equal to L^{-l} and hence is a $U_q \mathfrak{sl}_N$ -invariant.*

Proof. It suffices to apply remark 8.2 and the commutation relation

$$\mathcal{J} \otimes \mathcal{J}(\chi_k)(t \otimes t^*) = q^{-2k}(t \otimes t^*)\mathcal{J} \otimes \mathcal{J}(\chi_k),$$

which follows from

$$\mathcal{J}(z_a^\alpha)t = qt\mathcal{J}(z_a^\alpha), \quad a = 1, \dots, n; \alpha = 1, \dots, m. \quad \blacksquare$$

Consider the integral operator $\hat{K}_l : D(\mathbb{U})_q \rightarrow D(\mathbb{U})'_q$; $\hat{K}_l : f \mapsto \text{id} \otimes \nu(K_l(1 \otimes fy^l))$, with $l \in -\mathbb{N}$, and $\nu : D(\mathbb{U})_q \rightarrow \mathbb{C}$ being an invariant integral.

Proposition 8.5 *For all $l \in -\mathbb{N}$, $a \in U_q \mathfrak{sl}_N$ one has the equality of operators from $D(\mathbb{U})_q$ into itself*

$$\pi_l(a)\hat{K}_l = \hat{K}_l\pi_l(a).$$

Proof. Consider the integral operator $\hat{\mathcal{K}}_l$ on the quantum principal homogeneous space determined by its kernel $\mathcal{K}_l = (t \otimes t^*)^{-l}\mathcal{J} \otimes \mathcal{J}(K_l)$. We need also an extension by a continuity of the map $\mathcal{J}_l : \text{Pol}(\text{Mat}_{mn})_q \rightarrow \text{Pol}(\tilde{X})_{q,x}$; $\mathcal{J}_l : f \mapsto \mathcal{J}(f)t^{-l}$ onto the space $\text{Fun}(\mathbb{U})_q = \text{Pol}(\text{Mat}_{mn})_q + D(\mathbb{U})_q$.

It suffices to prove the relations

$$\pi(a)\hat{\mathcal{K}}_l f = \hat{\mathcal{K}}_l \pi(a)f, \quad a \in U_q \mathfrak{sl}_N, f \in D(\tilde{X})_q,$$

$$\pi_l(a) = \mathcal{I}_l^{-1} \pi(a) \mathcal{I}_l, \quad a \in U_q \mathfrak{sl}_N,$$

$$\widehat{K}_l = \mathcal{I}_l^{-1} \widehat{\mathcal{K}}_l \mathcal{I}_l.$$

The first of those follows from the invariance of the kernel \mathcal{K}_l and invariance of the integral involved when constructing the operator $\widehat{\mathcal{K}}_l$. The second relation is a consequence of lemma 6.3 and a continuity argument. The latter equality follows from the fact that for all $\psi \in \text{Pol}(\text{Mat}_{mn})_q$ and $f \in D(\mathbb{U})_q$ one has $\int_{\widetilde{X}_q} (t^*)^{-l} \mathcal{I}(\psi) \mathcal{I}(f) t^{-l} d\nu = \text{const} \int_{\mathbb{U}_q} \psi f d\nu_l$.

This relation is proved as follows:

$$\begin{aligned} \int_{\widetilde{X}_q} (t^*)^{-l} \mathcal{I}(\psi) \mathcal{I}(f) t^{-l} d\nu &= \int_{\widetilde{X}_q} (t^*)^{-l} \mathcal{I}(\psi f) t^{-l} d\nu = \int_{\widetilde{X}_q} \mathcal{I}(\psi f) t^{-l} (t^*)^{-l} d\nu = \int_{\widetilde{X}_q} \mathcal{I}(\psi f) (tt^*)^{-l} d\nu = \\ &= \int_{\widetilde{X}_q} \mathcal{I}(\psi f y^l) d\nu = \int_{\mathbb{U}_q} \psi f y^l d\nu = \text{const} \int_{\mathbb{U}_q} \psi f d\nu_l. \end{aligned}$$

(The above argument applies the relations $tt^* = t^*t$ and $\int_{\widetilde{X}_q} t^* f d\nu = \int_{\widetilde{X}_q} f t^* d\nu$, $f \in D(\widetilde{X})_q$,

together with theorem 2.5. The latter relation follows from the special case $f \in D(\widehat{X})_q$, and hence from even more special case $f = \varphi t$, $\varphi \in D(X)_q$. Now for $\varphi \in D(X)_q$ one has $\int_{\widetilde{X}_q} t^* \varphi t d\nu = \int_{\widetilde{X}_q} \varphi t t^* d\nu$, as one can easily deduce from the explicit formula for invariant integral.) ■

Now pass from the special case $l \in -\mathbb{N}$ to the general case via 'analytic continuation'.

We need in the sequel some integral operators whose kernels depend on a parameter u .

The term 'polynomial kernels' will stand for the formal series $K(u) = \sum_{i,j=0}^{\infty} K(u)^{(i,j)}$

whose terms belong to the $\mathbb{C}[u]$ -module $\mathbb{C}[\text{Mat}_{mn}]_{q,i}^{\text{op}} \otimes \mathbb{C}[\text{Mat}_{mn}]_{q,-j} \otimes \mathbb{C}[u]$. The vector space of all polynomial kernels carries a natural structure of algebra over $\mathbb{C}[u]$. For any $u_0 \in \mathbb{C}$ one has a well defined homomorphism $K(u) \mapsto K(u_0)$ of this algebra into $\mathbb{C}[[\text{Mat}_{mn} \times \overline{\text{Mat}}_{mn}]]_q$.

Proposition 8.6 *There exists a unique polynomial kernel $K(u)$ such that for all $l \in -\mathbb{N}$, $K(q^{2l}) = K_l$.*

Proof. The uniqueness of $K(u)$ is obvious. It follows from (8.3) that for $l = -1, -2, -3, \dots$,

$$K_l = \left(1 + \sum_{k=1}^m (-q^{2l})^k \chi_k \right) \cdot K_{l+1}. \quad (8.4)$$

Remove the parentheses and reduce the right hand side of (8.3) without using any commutation relations (note that all the commutation relations between the generators

of $\mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}}$ and $\mathbb{C}[\overline{\text{Mat}}_{mn}]_q$ are homogeneous of order two. It suffices to prove that for each word over the alphabet $\{\chi_1, \chi_2, \dots, \chi_m\}$ the associated coefficient is a polynomial of $u = q^{2l}$. To do this, observe that the coefficient at the void word (free term) is 1, and the polynomial nature of other coefficients is deducible via (8.4) using an induction argument with respect to the length of the word. \blacksquare

Define the kernels $K_l \in \mathbb{C}[[\text{Mat}_{mn} \times \overline{\text{Mat}}_{mn}]]_q$ for all $l \in \mathbb{C}$ by $K_l = K(q^{2l})$, with $K(u)$ being the polynomial kernel whose existence and uniqueness have just been proved.

Corollary 8.7 *For $l \in \mathbb{N}$,*

$$K_l = \left(1 + \sum_{k=1}^m (-q^{2(l-1)})^k \chi_k\right)^{-1} \cdot \left(1 + \sum_{k=1}^m (-q^{2(l-2)})^k \chi_k\right)^{-1} \cdots \left(1 + \sum_{k=1}^m (-1)^k \chi_k\right)^{-1}. \quad (8.5)$$

Proof. The validity of this relation for $l \in \mathbb{C}$ follows from the polynomial nature of $K(u)$ and the validity of (8.4) for $l \in \{-1, -2, -3, \dots\}$. \blacksquare

By a virtue of proposition 4.4, for any $l \in \mathbb{C}$ one has a well defined operator with kernel K_l :

$$\widehat{K}_l : D(\mathbb{U})_q \rightarrow D(\mathbb{U})'_q; \quad \widehat{K}_l : \text{id} \otimes \nu(K_l(1 \otimes f)y^l).$$

Proposition 8.5 admits the following generalization.

Proposition 8.8 *For all $l \in \mathbb{C}$, $a \in U_q \mathfrak{sl}_N$, one has the equality of operators from $D(\mathbb{U})_q$ to $D(\mathbb{U})'_q$:*

$$\pi_l(a) \widehat{K}_l = \widehat{K}_l \pi_l(a).$$

Proof. It suffices to obtain the relation

$$\int_{\mathbb{U}_q} f_2(\pi_l(a) \widehat{K}_l f_1) d\nu = \int_{\mathbb{U}_q} f_2(\widehat{K}_l \pi_l(a) f_1) d\nu \quad (8.6)$$

for all $f_1, f_2 \in D(\mathbb{U})_q$, $a \in U_q \mathfrak{sl}_N$, $l \in \mathbb{C}$. By a virtue of proposition 8.5 this relation is valid for all $l \in -\mathbb{N}$. What remains is to prove that both hand sides of 8.6 are Laurent polynomials of the indeterminate $v = q^l$. As one can observe from (3.1), (3.3), it suffices to prove that the integrands in (8.6) are Laurent polynomials of $v = q^l$. (The function $f(v)$ with values in $D(\mathbb{U})_q$ is called a Laurent polynomial if all the operator valued functions $\Theta_{ij}(f(v))$, $i, j \in \mathbb{Z}_+$, are Laurent polynomials (see section 4)). The polynomial nature of the integrands in (8.6) now follows from (6.2) – (6.4) and proposition 4.4. The latter proposition implies that the formal series $K(u) = \sum_{i,j=0}^{\infty} K^{(i,j)}$ which is implicit in both

hand sides of (8.6) can be replaced by a finite sum $\sum_{i,j=0}^M K^{(i,j)}$, with $M = M(f_1, f_2) \in \mathbb{N}$.

Now what remains is to remind that the operator valued functions $K^{(i,j)}(u)$, $u = q^{2l}$, are polynomials. \blacksquare

9 q-analogues of Bergman kernels

In section 8 the kernels $K_\lambda \in \mathbb{C}[[\text{Mat}_{mn} \times \overline{\text{Mat}_{mn}}]]_q$ have been defined in the special case $\lambda \in \mathbb{Z}$ an explicit formula for K_λ was presented in section 8; the general case $\lambda \in \mathbb{C}$ is to be considered in section 10.

We are going to show that the orthogonal projections P_λ onto Hardy-Bergman subspaces are integral operators with kernels K_λ ; in different terms, these kernels are q-analogues of Bergman kernels. (see [3]).

Theorem 9.1 *For all $\lambda > N - 1$, $f \in D(\mathbb{U})_q$ one has*

$$P_\lambda f = (\text{id} \otimes \nu_\lambda)(K_\lambda(1 \otimes f)). \quad (9.1)$$

Proof. Consider the special case $f = f_0$. Evidently,

$$(\text{id} \otimes \nu_\lambda)(K_\lambda(1 \otimes f)) = \left(\int_{\mathbb{U}_q} f_0 d\nu_\lambda \right) \cdot 1. \quad (9.2)$$

Prove that

$$P_\lambda f_0 = \left(\int_{\mathbb{U}_q} f_0 d\nu_\lambda \right) \cdot 1. \quad (9.3)$$

In fact, it follows from proposition 6.6 that the element $P_\lambda f_0 \in \mathbb{C}[\text{Mat}_{mn}]_q$ is subject to the relations $H_0(P_\lambda f_0) = P_\lambda(H_0 f_0) = 0$. Hence $P_\lambda f_0 = \text{const}(\lambda) \cdot 1$. On the other hand, $\|1\|_\lambda = 1$ by a virtue of proposition 3.4. What remains is to use the fact that P_λ is an orthogonal projection in $L^2(d\nu_\lambda)_q$: $\text{const}(\lambda) = \frac{1}{\|1\|_\lambda} \int_{\mathbb{U}_q} 1 \cdot f_0 d\nu_\lambda = \int_{\mathbb{U}_q} f_0 d\nu_\lambda$. Now (9.2),

(9.3) imply (9.1) for $f = f_0$.

Our next step is to pass from the special case $f = f_0$ to the general case. Let L_λ be the subspace of all those $f \in D(\mathbb{U})_q$ which satisfy (9.1). Prove that $D(\mathbb{U})_q \subset L_\lambda$.

We know that $f_0 \in L_\lambda$. By a virtue of propositions 6.6 and 8.8, for all $a \in U_q \mathfrak{sl}_N$ one has $\pi_\lambda(a)L_\lambda \subset L_\lambda$. Hence $\{\pi_\lambda(a)f_0 \mid a \in U_q \mathfrak{sl}_N\} \subset L_\lambda$. Apply proposition 7.4 to complete the proof. \blacksquare

Note that the measure $d\mu = d\nu_N$ is a q-analogue of the Lebesgue measure in the matrix ball, and the kernel

$$K_N = \left(1 + \sum_{k=1}^m (-q^{2(N-1)})^k \chi_k \right)^{-1} \left(1 + \sum_{k=1}^m (-q^{2(N-2)})^k \chi_k \right)^{-1} \cdots \left(1 + \sum_{k=1}^m (-1)^k \chi_k \right)^{-1}$$

is a q-analogue of the ordinary Bergman kernel.

REMARK 9.2. The notation $\mathbf{z} = (z_a^\alpha)_{a=1, \dots, n; \alpha=1, \dots, m}$, $\boldsymbol{\zeta} = (\zeta_a^\alpha)_{a=1, \dots, n; \alpha=1, \dots, m}$, allow one to rewrite (9.1) in a more appropriate form as

$$P_\lambda = \int_{\mathbb{U}_q} K_\lambda(\mathbf{z}, \boldsymbol{\zeta}) f(\boldsymbol{\zeta}) d\nu_\lambda(\boldsymbol{\zeta}).$$

10 Pairwise commuting kernels

Our purpose is to prove the following statements.

Lemma 10.1 *In the algebra $\mathbb{C}[SU_m]_q^{\text{op}} \otimes \mathbb{C}[SU_m]_q$ the elements*

$$\sum_{\substack{J', J'' \subset \{1, \dots, m\} \\ \text{card}(J') = \text{card}(J'') = k}} z^{\wedge k}_{J''} \otimes \left(z^{\wedge k}_{J'} \right)^*, \quad k = 1, 2, \dots, m-1 \quad (10.1)$$

are pairwise commuting.

Lemma 10.2 *In the algebra $\mathbb{C}[\text{Mat}_{mm}]_q^{\text{op}} \otimes \mathbb{C}[\overline{\text{Mat}}_{mm}]_q$ the elements χ_k , $k = 1, \dots, m$, given by (8.2), are pairwise commuting.*

Proposition 10.3 *In the algebra $\mathbb{C}[[\text{Mat}_{mn} \times \overline{\text{Mat}}_{mn}]]_q$ the elements χ_k , $k = 1, \dots, m$, given by (8.2), are pairwise commuting, and*

$$K_\lambda = \prod_{j=0}^{\infty} \left(1 + \sum_{k=1}^m (-q^{2(\lambda+j)})^k \chi_k \right) \left(\prod_{j=0}^{\infty} \left(1 + \sum_{k=1}^m (-q^{2j})^k \chi_k \right) \right)^{-1} \quad (10.2)$$

for all $\lambda \in \mathbb{C}$.

REMARK 10.4. The relations (8.3) and (8.5) are special cases of (10.2).

Proof of lemma 10.1. Apply the embedding of Hopf \ast -algebras $\mathbb{C}[SU_m]_q \hookrightarrow (U_q \mathfrak{su}_m)^*$ (see [1]). It sends the elements $z_a^\alpha \in \mathbb{C}[SU_m]_q$ to matrix elements of operators of the vector representation π of the \ast -algebra $U_q \mathfrak{su}_m$ in the orthonormal basis of weight vectors. The minors $z^{\wedge k}_{J''}$ for $k > 1$ could also be treated in terms of exterior powers $\pi^{\wedge k}$ of the vector representation (see [4]). Hence one has (see [1, 5]):

$$\Delta(z^{\wedge k}_{J''}) = \sum_{\text{card}(J)=k} z^{\wedge k}_{J'} \otimes z^{\wedge k}_{J''}, \quad (10.3)$$

$$\left(z^{\wedge k}_{J'} \right)^* = S \left(z^{\wedge k}_{J''} \right), \quad (10.4)$$

with $J, J', J'' \subset \{1, 2, \dots, m\}$, S and Δ being antipode and comultiplication of the Hopf algebra $\mathbb{C}[SU_m]_q$. By a virtue of (10.3), (10.4), the elements (10.1) can be rewritten as

$$\left((S \otimes \text{id}) \Delta \left(\sum_{\substack{J \subset \{1, 2, \dots, m\} \\ \text{card}(J)=k}} z^{\wedge k}_J \right) \right)^{\ast \otimes \ast} . \text{ Since the linear map}$$

$$\mathbb{C}[SU_m]_q \rightarrow \mathbb{C}[SU_m]_q^{\text{op}} \otimes \mathbb{C}[SU_m]_q, \quad f \mapsto (S \otimes \text{id}) \Delta(f)$$

is a homomorphism of algebras, it suffices to prove the pairwise commutativity of the elements $\sum_{\substack{J \subset \{1,2,\dots,m\} \\ \text{card}(J)=k}} z^{\wedge k, J} \in \mathbb{C}[SU_m]_q$. What remains is to apply the equivalence of the

representations $\pi^{\wedge k_1} \otimes \pi^{\wedge k_2}$ and $\pi^{\wedge k_2} \otimes \pi^{\wedge k_1}$ for all $k_1, k_2 \in \mathbb{Z}_+$. \blacksquare

Proof of lemma 10.2 can be obtained via replacing the quantum groups SU_m with the quantum group U_m in the proof of lemma 10.1. Specifically, consider the Hopf \ast -algebra $\mathbb{C}[H]$ with the standard comultiplication Δ and involution \star : $\Delta(H) = H \otimes 1 + 1 \otimes H$, $H^\star = H$. Our definition implies $U_q \mathfrak{u}_m = U_q \mathfrak{su}_m \otimes \mathbb{C}[H]$. What remains is to demonstrate an embedding $\mathbb{C}[\text{Mat}_{mm}]_q \hookrightarrow (U_q \mathfrak{u}_m)^\star$.

For that, consider the algebra $\mathbb{C}[GL_m]_q$ being a localization of $\mathbb{C}[\text{Mat}_{mm}]_q$ with respect to the multiplicative system $(\det_q \mathbf{z})^\mathbb{N}$, where, as in an ordinary setting,

$$\det_q \mathbf{z} = \sum_{s \in S_m} (-q)^{l(s)} z_{s(1)}^1 z_{s(2)}^2 \cdots z_{s(m)}^m.$$

The algebra $\mathbb{C}[GL_m]_q$ is equipped with a structure of Hopf algebra in a standard way and is called an algebra of functions on the quantum group GL_m . Equip this Hopf algebra with an involution:

$$(z_a^\alpha)^\star = (-q)^{a-\alpha} (\det_q \mathbf{z})^{-1} \det_q (\mathbf{z}_a^\alpha),$$

with \mathbf{z}_a^α being the matrix derivable from \mathbf{z} by discarding the line α and the column a . The resulting Hopf \ast -algebra $\mathbb{C}[U_m]_q = (\mathbb{C}[GL_m]_q, \star)$ will be called an algebra of regular functions on the quantum group U_m . Now we have an embedding of algebras $\mathbb{C}[\text{Mat}_{mm}]_q \hookrightarrow \mathbb{C}[U_m]_q$ and an embedding $\mathbb{C}[U_m]_q \hookrightarrow (U_q \mathfrak{u}_m)^\star$. That is,

$$\mathbb{C}[\text{Mat}_{mm}]_q \hookrightarrow \mathbb{C}[U_m]_q \hookrightarrow (U_q \mathfrak{u}_m)^\star.$$

Consider the involutive algebra $F = \mathbb{C}[SU_m]_q \otimes \mathbb{C}[u, u^{-1}]$, $u^\star \stackrel{\text{def}}{=} u^{-1}$. We need embeddings of algebras

$i_1 : \mathbb{C}[\text{Mat}_{mm}]_q^{\text{op}} \hookrightarrow F^{\text{op}}$, $i_2 : \mathbb{C}[\overline{\text{Mat}}_{mm}]_q \hookrightarrow F$, $i : \mathbb{C}[\text{Mat}_{mm}]_q^{\text{op}} \otimes \mathbb{C}[\overline{\text{Mat}}_{mm}]_q \hookrightarrow F^{\text{op}} \otimes F$, given by

$$i_1(z_a^\alpha) = z_a^\alpha \otimes u, \quad i_2((z_a^\alpha)^\star) = (z_a^\alpha)^\star \otimes u^\star, \quad i(f_1 \otimes f_2) = i_1(f_1) \otimes i_2(f_2).$$

It follows from the definitions that

$$i(\chi_k) = \sum_{\substack{J', J'' \subset \{1, \dots, m\} \\ \text{card}(J') = \text{card}(J'') = k}} (z^{\wedge k, J'} \otimes u^k) \otimes ((z^{\wedge k, J''})^\star \otimes u^{-k}), \quad k \neq m,$$

$$i(\chi_m) = (1 \otimes u^m) \otimes (1 \otimes u^{-m}).$$

Thus we deduce from lemma 10.1 that

$$i(\chi_{k_1}) i(\chi_{k_2}) = i(\chi_{k_2}) i(\chi_{k_1}), \quad k_1, k_2 = 1, \dots, m. \quad \blacksquare$$

Proof of proposition 10.3. Show that the first statement reduces to the special case $m = n$ which was considered in lemma 10.2. Consider the homomorphisms of algebras

$$j_1 : \mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}} \rightarrow \mathbb{C}[\text{Mat}_{mm}]_q^{\text{op}}, \quad j_2 : \mathbb{C}[\overline{\text{Mat}}_{mn}]_q \rightarrow \mathbb{C}[\overline{\text{Mat}}_{mm}]_q,$$

$$j : \mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}} \otimes \mathbb{C}[\overline{\text{Mat}}_{mn}]_q \rightarrow \mathbb{C}[\text{Mat}_{mm}]_q^{\text{op}} \otimes \mathbb{C}[\overline{\text{Mat}}_{mm}]_q$$

given by

$$j_1(z_a^\alpha) = \begin{cases} z_{a-(n-m)}^\alpha, & a > n-m; \\ 0, & a \leq n-m; \end{cases} \quad j_2((z_a^\alpha)^*) = \begin{cases} (z_{a-(n-m)}^\alpha)^*, & a > n-m; \\ 0, & a \leq n-m; \end{cases}$$

$$j(f_1 \otimes f_2) = j_1(f_1) \otimes j_2(f_2).$$

It suffices to prove the injectivity of the restriction of j onto the subalgebra F_0 generated by χ_k , $k = 1, 2, \dots, m$.

Let $\psi \in F_0$, $j(\psi) = 0$. Choose $\lambda > N - 1$ and consider the integral operator with kernel ψ :

$$\widehat{\psi} : \mathbb{C}[\text{Mat}_{mn}]_q \rightarrow \mathbb{C}[\text{Mat}_{mn}]_q; \quad \widehat{\psi} : f \mapsto \text{id} \otimes \nu_\lambda(\psi(1 \otimes f)).$$

Note that instead of the relation $\psi = 0$ we may prove $\widehat{\psi} = 0$ since the scalar product $(f_1, f_2)_\lambda = \int_{\mathbb{U}_q} f_2^* f_1 d\nu_\lambda$ in the vector space $\mathbb{C}[\text{Mat}_{mn}]_q$ is nondegenerate.

Remind that $U_q \mathfrak{s}(\mathbf{u}_n \times \mathbf{u}_m) \subset U_q \mathfrak{s}\mathbf{u}_{nm}$ is a Hopf $*$ -subalgebra generated by $K_n^{\pm 1}$, $\{E_j, F_j, K_j^{\pm 1}\}_{j \neq n}$. It is easy to show (see [9]) that the $U_q \mathfrak{s}(\mathbf{u}_n \times \mathbf{u}_m)$ -invariance of χ_k , $k = 1, \dots, m$, implies the $U_q \mathfrak{s}(\mathbf{u}_n \times \mathbf{u}_m)$ -invariance of $\psi \in F_0$. Furthermore, the $U_q \mathfrak{s}(\mathbf{u}_n \times \mathbf{u}_m)$ -invariance of y implies the $U_q \mathfrak{s}(\mathbf{u}_n \times \mathbf{u}_m)$ -invariance of the integral ν_λ . Hence the linear map $\widehat{\psi}$ is a morphism of $U_q \mathfrak{s}(\mathbf{u}_n \times \mathbf{u}_m)$ -modules.

If $\psi \in F_0$ and $j(\psi) = 0$, then one readily deduces that $\widehat{\psi}$ is zero on the subalgebra generated by z_a^α , $a > n - m$. (In fact, if $\psi \in F_0$, then $j(\psi) = 0$ is equivalent to $\text{id} \otimes j_2(\psi) = 0$. Hence it suffices to prove the relation $\int_{\mathbb{U}_q} f^* \varphi d\nu_\lambda = 0$ for any element φ of the subalgebra

generated by z_a^α , $a > n - m$, and any element f such that $j_2(f^*) = 0$. One can assume without loss of generality that

$$f = (z_1^1)^{k_{11}} (z_1^2)^{k_{21}} \dots (z_1^m)^{k_{m1}} (z_2^1)^{k_{12}} (z_2^2)^{k_{22}} \dots (z_2^m)^{k_{m2}} \dots (z_n^m)^{k_{mn}},$$

$$\psi = (z_{n-m+1}^1)^{l_{1,n-m+1}} (z_{n-m+1}^2)^{l_{2,n-m+1}} \dots (z_{n-m+1}^m)^{l_{m,n-m+1}} \dots (z_n^m)^{l_{m,n}},$$

with $k_{i'j'} \neq 0$ for some $1 \leq i' \leq m$, $1 \leq j' \leq n - m$. However, in this case the assumption $\int_{\mathbb{U}_q} f^* \varphi d\nu_\lambda \neq 0$ leads to a contradiction since for all $j = 1, 2, \dots, n - 1$ one

$$\text{has } H_j(f^* \varphi) = 0, \left(\sum_{i=1}^m k_{ij} - \sum_{i=1}^m l_{ij} \right) - \left(\sum_{i=1}^m k_{i,j+1} - \sum_{i=1}^m l_{i,j+1} \right) = 0,^2 \text{ and that } H_0(f^* \varphi) = 0, \\ \sum_{j=1}^{n-1} \left(\sum_{i=1}^m k_{ij} - \sum_{i=1}^m l_{ij} \right) = 0.)$$

The morphism of $U_q \mathfrak{s}(\mathbf{u}_n \times \mathbf{u}_m)$ -modules $\widehat{\psi} : \mathbb{C}[\text{Mat}_{mn}]_q \rightarrow \mathbb{C}[\text{Mat}_{mn}]_q$ sends to zero all the generators

$$f_{j_1 j_2 \dots j_m} = \prod_{k=1}^m \left(z^{\wedge_k \{1, 2, \dots, k\}_{n-k+1, n-k+2, \dots, n}} \right)^{j_k}$$

of the $U_q \mathfrak{s}(\mathbf{u}_n \times \mathbf{u}_m)$ -module $\mathbb{C}[\text{Mat}_{mn}]_q$. Hence $\widehat{\psi} = 0$, and thus the pairwise commutativity of χ_k , $k = 1, \dots, m$, is proved.

²We assume that $l_{ij} = 0$ for $j \leq n - m$.

What remains is to obtain the relation (10.2). Just the same argument as that used in the proof of proposition 8.6, allows one to establish that the kernel $\prod_{j=0}^{\infty} \left(1 + \sum_{k=1}^m (-uq^{2j})^k \chi_k\right)$ is polynomial (see section 8). Hence

$$K(u) = \prod_{j=0}^{\infty} \left(1 + \sum_{k=1}^m (-uq^{2j})^k \chi_k\right) \cdot \left(\prod_{j=0}^{\infty} \left(1 + \sum_{k=1}^m (-q^{2j})^k \chi_k\right)\right)^{-1},$$

since the kernels in both hand sides are polynomials and coincide with kernels (8.3) as $u = q^{2l}$, $l \in -\mathbb{N}$. To conclude, use the definition of K_λ : $K_\lambda = K(q^{2\lambda})$. ■

Appendix. Boundedness of the quantum matrix ball

Consider the faithful $*$ -representation Π of $\text{Pol}(\text{Mat}_{mn})_q$ in the *pre-Hilbert* space $\tilde{\mathcal{H}}$, described in [7, appendix 2]. We use here the norm of the $m \times n$ matrix with entries in $\text{End } \tilde{\mathcal{H}}$ defined as a norm of the associated linear map $\bigoplus_{a=1}^n \tilde{\mathcal{H}} \rightarrow \bigoplus_{\alpha=1}^m \tilde{\mathcal{H}}$.

Proposition A.1. *Let Z and $\Pi(Z)$ be the matrices $(z_{\alpha a})_{\alpha=1,\dots,m, a=1,\dots,n}$ and $(\Pi(z_{\alpha a}))_{\alpha=1,\dots,m, a=1,\dots,n}$, respectively. Then $\|\Pi(Z)\| \leq 1$.*

Proof. Let S be the antipode of the Hopf algebra $\mathbb{C}[SL_N]_q$; its action on the generators is given by a well known formula (see [1]):

$$S(t_{a\beta}) = (-q)^{a-\beta} \det_q(T_{\beta a}), \quad a, \beta = 1, \dots, N. \quad (\text{A.1})$$

(The matrix $T_{\beta a}$ in (A.1) is derived from $T = (t_{ij})_{i,j=1,\dots,N}$ by discarding line β and column a .) Hence

$$\sum_{a=1}^N (-q)^{a-\beta} t_{\alpha a} \det_q(T_{\beta a}) = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, N,$$

or, equivalently,

$$-\sum_{c=1}^n t_{\alpha c} t_{\beta c}^* + \sum_{\gamma=1}^m t_{\alpha, n+\gamma} t_{\beta, n+\gamma}^* = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, m,$$

with $*$ being the involution in $\text{Pol}(\tilde{X})_q$ (see [7]). After introducing a notation

$$\begin{aligned} T_{11} &= (t_{\alpha a})_{\alpha=1,\dots,m, a=1,\dots,n}; & T_{12} &= (t_{\alpha, n+\beta})_{\alpha, \beta=1,\dots,m}, \\ T_{11}^* &= (t_{a\alpha}^*)_{\alpha=1,\dots,m, a=1,\dots,n}; & T_{12}^* &= (t_{n+\beta, \alpha}^*)_{\alpha, \beta=1,\dots,m}, \end{aligned}$$

we get

$$-T_{11} T_{11}^* + T_{12} T_{12}^* = I. \quad (\text{A.2})$$

It follows from (A.2) and (2.11) that $\mathcal{I}(I - ZZ^*) = T_{12}^{-1} (T_{12}^{-1})^*$.

Apply the representation $\tilde{\Pi}$ (see [7]) to both parts of the above relation. By a virtue of $\Pi = \tilde{\Pi} \mathcal{I}$ we obtain $\Pi(I - ZZ^*) = \tilde{\Pi}(T_{12}^{-1}) \tilde{\Pi}(T_{12}^{-1})^* \geq 0$. Hence $\Pi(Z) \Pi(Z)^* \leq I$, $\|\Pi(Z)\| = \|\Pi(Z^*)\| \leq 1$. ■

Now a passage from $\tilde{\mathcal{H}}$ to its completion allows one to obtain a representation of the $*$ -algebra $\text{Pol}(\text{Mat}_{mn})_q$ by bounded operators in a Hilbert space: $\|\Pi(z_a^\alpha)\| \leq 1$, $a = 1, \dots, n$, $\alpha = 1, \dots, m$.

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